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Vicious walkers in a potential

Alan J Bray and Karen Winkler

Department of Physics and Astronomy, University of Manchester, Manchester M13 9PL, UK

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Abstract
We consider N vicious walkers moving in one dimension in a one-body potential $v(x)$. Using the backward Fokker–Planck equation we derive exact results for the asymptotic form of the survival probability $Q(\mathbf{x}, t)$ of vicious walkers initially located at $(x_1, \ldots, x_N) = \mathbf{x}$, when $v(x)$ is an arbitrary attractive potential. Explicit results are given for a square-well potential with absorbing or reflecting boundary conditions at the walls, and for a harmonic potential with an absorbing or reflecting boundary at the origin and the walkers starting on the positive half line. By mapping the problem of N vicious walkers in zero potential onto the harmonic potential problem, we rederive the results by Fisher (1984 *J. Stat. Phys.* **34** 667) and Krattenthaler *et al* (2000 *J. Phys. A: Math. Gen.* **33** 8835) respectively for vicious walkers on an infinite line and on a semi-infinite line with an absorbing wall at the origin. This mapping also gives a new result for vicious walkers on a semi-infinite line with a reflecting boundary at the origin: $Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{2}}$.

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1. Introduction

Consider N symmetric random walkers which annihilate on meeting each other but otherwise do not interact. This concept of short-ranged interacting random walkers was introduced by M E Fisher as the vicious walkers model [1].

One of the main properties of interest in this model is the survival probability $Q(\mathbf{x}, t)$ that none of the N vicious walkers with initial position coordinates $(x_1, x_2, \ldots, x_N) = \mathbf{x}$ has met another up to time t, i.e. none of them has been annihilated up to time t.

Fisher and Huse [1, 2] determined the survival probability for N vicious walkers moving on an infinite line. For large times, $Q(\mathbf{x}, t)$ decays as a power:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{4}}. \tag{1}$$

Interesting results also arise when further conditions are imposed on the movement of the vicious walkers by the use of absorbing or reflecting walls, where all walkers are initially located on the same side of the boundary (the case where there are walkers on both sides decouples into two independent problems). While Fisher [1] found the survival probability problem of vicious walkers with an absorbing boundary at the origin only for $N = 2$, Krattenthaler *et al* [3] were able to determine exact asymptotic forms for N vicious walkers starting from equi-spaced lattice points:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N^2}{2}}. \tag{2}$$

By evaluating the scaling limit Katori and Tanemura [4] showed that this asymptotic behaviour holds for arbitrary initial conditions on a continuous line.

In this paper we introduce the interesting problem of N vicious walkers moving in an attractive one-body potential $v(x)$, i.e. the full potential function has the separable form $V(\mathbf{x}) = \sum_{i=1}^{N} v(x_i)$. Treating both time and space as continuous, we investigate the survival probability of N vicious walkers with equal diffusion constants D. The equation of motion for walker i is taken to be

$$\dot{x}_i = -\frac{\partial V}{\partial x_i} + \eta_i(t) \tag{3}$$

where the Langevin noise $\eta_i(t)$ is a Gaussian white noise with mean zero and correlator

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t - t'). \tag{4}$$

For a square-well potential of width L we consider three different combinations of absorbing and reflecting walls and find an exponential decay for the survival probability of the general form $Q(\mathbf{x}, t) \sim \mathrm{e}^{-\theta_N t}$. For two reflecting walls the exponent θ_N is determined to be

$$\theta_N^{RR} = D\frac{\pi^2}{L^2}\frac{N(N-1)(2N-1)}{6}. \tag{5}$$

In the case of one reflecting and one absorbing wall we obtain

$$\theta_N^{RA} = D\frac{\pi^2}{L^2}\frac{N(2N+1)(2N-1)}{12} \tag{6}$$

while for two absorbing walls the exponent of the asymptotic decay is

$$\theta_N^{AA} = D\frac{\pi^2}{L^2}\frac{N(N+1)(2N+1)}{6}. \tag{7}$$

An interesting potential which turns out to be a powerful tool is the problem of N vicious walkers in the harmonic potential $V(\mathbf{x}) = \frac{a}{2}\mathbf{x}^2$. The asymptotic behaviour for large times is determined to be an exponential decay independent of the diffusion constant:

$$\theta_N = \frac{N(N-1)}{2}a. \tag{8}$$

This result also provides a mechanism to determine the survival probability of N vicious walkers on an infinite line in a simple way. By mapping the zero-potential problem to the harmonic potential problem, we derive Fisher's result [1] and also the result by Krattenthaler *et al* [3] with an absorbing wall at the origin. Furthermore we are able to obtain, to our knowledge, a new result for the survival probability of N vicious walkers on a semi-infinite line with a reflecting boundary at the origin, which decays as

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{2}}. \tag{9}$$

The outline of the paper is as follows. In section 2 we present the method for a general one-body potential $v(x)$, while in section 3 we give explicit results for square-well and harmonic potentials. In section 4 we revisit the case of zero potential, obtaining the known results, and a new result for a system with a reflecting boundary, through a transformation to the harmonic problem. Section 5 is a short conclusion.

2. The method

The dynamics of a random walker, with position coordinate x_i, moving in a potential $V(\mathbf{x})$ is described, in continuous space and time, by the Langevin equation (3) with noise correlator (4).

The probability $Q(\mathbf{x}, t)$ that all N vicious walkers, $i = 1, \ldots, N$, have survived up to time t, given that they *started* at $\{x_i\}$, satisfies the corresponding backward Fokker–Planck equation:

$$\frac{\partial Q(\mathbf{x}, t)}{\partial t} = D \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} Q(\mathbf{x}, t) - \sum_{i=1}^{N} \frac{\partial V(\mathbf{x})}{\partial x_i} \frac{\partial Q(\mathbf{x}, t)}{\partial x_i}. \tag{10}$$

For convenience we start by defining the survival probability $q(x_i, t)$ of just one random walker moving in a potential restricted by the imposed boundary conditions. This survival probability $q(x_i, t)$ satisfies the backward Fokker–Planck equation

$$\frac{\partial q(x_i, t)}{\partial t} = D \frac{\partial^2}{\partial x_i^2} q(x_i, t) - \frac{\mathrm{d}v(x_i)}{\mathrm{d}x_i} \frac{\partial q(x_i, t)}{\partial x_i} \tag{11}$$

where we have used the relation $V(\mathbf{x}) = \sum_i v(x_i)$ for a one-body potential. For any such potential, the backward Fokker–Planck equation (11) is separable in time and space. Let us call these separable solutions, i.e. the solutions of equation (11) satisfying the relevant boundary conditions, single-walker basis functions. They have the form $q_j(x_i, t) = u_j(x_i) \exp(-\lambda_j t)$, where λ_j is the decay rate associated with basis function j and these rates are ordered such that $\lambda_1 < \lambda_2 < \lambda_3 \cdots$.

For N non-interacting walkers moving in the same potential, the N-walker basis functions for the survival probability take the form of products of N single-walker functions, each with a different space variable x_i. Since, however, we are investigating vicious walkers the mutual annihilation property must be respected. Since two walkers die when arriving at the same x-coordinate, the boundary condition $Q(x_1, \ldots, x_n, t) = 0$ when $x_i = x_j$ for any $i \neq j$ must be respected. This property is ensured by constructing $Q(\mathbf{x}, t)$ using antisymmetric combinations of products of N single-walker functions, analogous to the antisymmetric construction of the wavefunction of fermions [1]. The N-walker basis functions of the vicious walker problem with N walkers have, therefore, the form

$$Q^{i_1,\ldots,i_N}(\mathbf{x}, t) = \det A^{i_1,\ldots,i_N} \tag{12}$$

where the elements of the $N \times N$ matrix A are given by

$$A_{nm}^{i_1,\ldots,i_N} = q_{i_n}(x_m, t). \tag{13}$$

The full solution $Q(\mathbf{x}, t)$ is a linear superposition of these basis functions with coefficients determined by the initial condition.

To solve the problem of N vicious walkers in an arbitrary potential, therefore, we need to find the single-walker basis functions $q_j(x_i, t)$ appropriate to the imposed boundary conditions. For an absorbing boundary at $x = a$ the functions $q_j(x_i, t)$ must satisfy

$$q_j(x_i = a, t) = 0. \tag{14}$$

For a reflecting boundary at $x = b$ the boundary condition for the backward Fokker–Planck equation is in general

$$\nabla Q(\mathbf{x}, t) \cdot \hat{\mathbf{n}} = 0 \tag{15}$$

where $\hat{\mathbf{n}}$ is normal to the reflecting boundary [5]. In the one-dimensional case, this expression implies

$$\left.\frac{\mathrm{d}q_j}{\mathrm{d}x_i}\right|_{x_i=b} = 0. \tag{16}$$

Clearly these boundary conditions are also satisfied by the functions $Q^{i_1,\ldots,i_N}(\mathbf{x}, t)$, since the latter is just an antisymmetrized product of single-walker basis functions.

Consider now the late-time limit, $t \to \infty$. Each antisymmetrized product in the expression for $Q(\mathbf{x}, t)$ contains N different relaxation factors $\exp(-\lambda_j t)$. The slowest-decaying term in the sum, therefore, is the term in which the relaxation rates are $\lambda_1, \lambda_2, \ldots, \lambda_N$. It follows that, asymptotically,

$$Q(\mathbf{x}, t) \propto \det B^{1,2,\ldots,N} \exp(-\theta_N t) \tag{17}$$

where $B^{1,2,\ldots,N}$ is just the $N \times N$ matrix with elements $B_{nm} = u_n(x_m)$ $(n, m = 1, \ldots, N)$, i.e. it is constructed using the N slowest-decaying single-walker basis functions, and the total decay rate is

$$\theta_N = \sum_{j=1}^{N} \lambda_j. \tag{18}$$

The following sections provide some applications of this general result.

3. Results for vicious walkers in a potential

In this section we discuss two examples of N vicious walkers in a potential and determine the decay of the survival probability $Q(\mathbf{x}, t)$.

3.1. The square-well potential

Consider a square-well potential which has two walls of infinite potential, one at the origin and the other at $x = L$, and vanishes between the walls. A vicious walker restricted to move between the walls satisfies the backward Fokker–Planck equation:

$$\frac{\partial q(x_i, t)}{\partial t} = D\frac{\partial^2 q(x_i, t)}{\partial x_i^2}. \tag{19}$$

This equation can be solved in general by separation of variables, which amounts in this case to writing the solution as a spatial Fourier series. Different solutions result from the various sets of boundary conditions imposed by the property of the walls.

3.1.1. Two reflecting walls. For two reflecting walls the spatial derivative of $q(x_i, t)$ must be zero at $x = 0$ and $x = L$. In this case, therefore $q(x_i, t)$ is given by Fourier cosine series with basis functions

$$q_n(x_i, t) = \exp\left(-\frac{n^2\pi^2 Dt}{L^2}\right)\cos\left(\frac{\pi}{L}nx_i\right) \qquad n = 0, 1, \ldots. \tag{20}$$

The survival probability is constructed as a superposition of antisymmetrized products of these basis functions:

$$
\begin{aligned}
Q(\mathbf{x}, t) &= \sum_{i_1} \cdots \sum_{i_N} C^{i_1,\ldots,i_N} \det A^{i_1,\ldots,i_N} \\
&= \sum_{i_1} \cdots \sum_{i_N} C^{i_1,\ldots,i_N} \exp\left(-\frac{\pi^2 Dt}{L^2} \sum_{n=1}^{N} i_n^2\right) \det B^{i_1,\ldots,i_N}
\end{aligned}
\tag{21}
$$

where

$$
B_{nm}^{i_1,\ldots,i_N} = \cos\left(\frac{\pi}{L} i_n x_m\right). \tag{22}
$$

To evaluate the long-time behaviour we keep only the N longest-lived modes, given by the N smallest values, $i = 0, 1, \ldots, N-1$ of i_n. Using $\sum_{i=0}^{N-1} i^2 = N(N-1)(2N-1)/6$ we obtain, for the asymptotic time dependence,

$$
Q(\mathbf{x}, t) \sim \exp\left(-\frac{\pi^2 Dt}{L^2} \frac{N(N-1)(2N-1)}{6}\right). \tag{23}
$$

3.1.2. One reflecting and one absorbing wall. For an absorbing wall at the origin and a reflecting wall at $x = L$ the boundary conditions are satisfied by a Fourier sine series with basis functions

$$
q_n(x_i, t) = \exp\left(-\frac{(2n+1)^2 \pi^2 Dt}{4L^2}\right) \sin\left(\frac{\pi}{2L}(2n+1) x_i\right) \qquad n = 0, 1, \ldots. \tag{24}
$$

Analogous to the preceding case the survival probability for all N vicious walkers is constructed and the asymptotic survival probability for large time is evaluated using $\sum_{i=0}^{N-1} (2i+1)^2 = N(2N+1)(2N-1)/3$ to give the asymptotic decay

$$
Q(\mathbf{x}, t) \sim \exp\left(-\frac{\pi^2 Dt}{L^2} \frac{N(2N+1)(2N-1)}{12}\right). \tag{25}
$$

3.1.3. Two absorbing walls. In the case of two absorbing walls the basis functions have to vanish at both $x = 0$ and $x = L$. A Fourier sine series is therefore appropriate, with basis functions

$$
q_n(x_i, t) = \exp\left(-\frac{n^2 \pi^2 Dt}{L^2}\right) \sin\left(\frac{\pi}{L} n x_i\right) \qquad n = 1, 2, \ldots. \tag{26}
$$

This is very similar to the result for two reflecting boundaries, except that the spatial functions are sines so the sum begins with $n = 1$. The large-time behaviour $Q(\mathbf{x}, t)$ is given by

$$
Q(\mathbf{x}, t) \sim \exp\left(-\frac{\pi^2 Dt}{L^2} \frac{N(N+1)(2N+1)}{6}\right). \tag{27}
$$

Before proceeding to the harmonic potential, we note that the inequalities $2N(N-1)(2N-1) < N(2N+1)(2N-1) < 2N(N+1)(2N+1)$, for all $N \geqslant 1$, imply that for a well of given size the decay is fastest with two absorbing boundaries and slowest with two reflecting boundaries, as is intuitively clear.

3.2. The harmonic potential

A harmonic potential $V(\mathbf{x}) = \frac{a}{2}\mathbf{x}^2$ is considered for which the backward Fokker–Planck equation for the one–walker basis function reads

$$\frac{\partial q(x_i,t)}{\partial t} = D\frac{\partial^2}{\partial x_i^2}q(x_i,t) - ax_i\frac{\partial q(x_i,t)}{\partial x_i}. \tag{28}$$

This equation can be transformed into an imaginary-time Schrödinger equation by the substitution $q(x_i,t) = \exp\left(ax_i^2/4D\right)\psi(x_i,t)$ to give

$$\frac{\partial \psi(x_i,t)}{\partial t} = D\frac{\partial^2}{\partial x_i^2}\psi(x_i,t) + \left(\frac{a}{2} - \frac{a^2x_i^2}{4D}\right)\psi(x_i,t). \tag{29}$$

This equation has solutions of the form $\psi(x_i,t) = \mathrm{e}^{-\lambda t}u(x_i)$, where $u(x_i)$ satisfies the ordinary differential equation

$$\left(D\frac{\mathrm{d}^2}{\mathrm{d}x_i^2} + \left(\frac{a}{2} - \frac{a^2x_i^2}{4D}\right)\right)u(x_i) = -\lambda u(x_i). \tag{30}$$

This equation is equivalent to the time-independent Schrödinger equation for the harmonic oscillator. The eigenvalues and eigenfunctions of this eigenvalue problem are well known: see for example a similar problem in [6]. The eigenfunctions have the form

$$u_n(x_i) = H_n\left(x_i\sqrt{\frac{a}{2D}}\right)\exp\left(-\frac{a}{4D}x_i^2\right) \tag{31}$$

where the functions $H_n(x)$ are the Hermite polynomials defined by

$$H_n(y) = (-1)^n\,\mathrm{e}^{y^2}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\,\mathrm{e}^{-y^2}. \tag{32}$$

The corresponding eigenvalues are $\lambda_n = na$, where $n = 0, 1, 2, \ldots.$ The original basis functions $q(x_i,t)$ are given by $q_n(x_i,t) = H_n\big(x_i\sqrt{\frac{a}{2D}}\big)\exp(-\lambda_n t)$.

Applying the antisymmetrization process to determine the survival probability of N vicious walkers in a harmonic potential we obtain the asymptotic time dependence:

$$Q(\mathbf{x},t) \sim \exp\left(-at\sum_{i=0}^{N-1} i\right) \tag{33}$$

giving

$$Q(\mathbf{x},t) \sim \exp\left(-at\frac{N(N-1)}{2}\right). \tag{34}$$

This approach is readily extended to the case where there is a reflecting or absorbing boundary at $x = 0$ and all the walkers start on the same side of the boundary (if there are walkers on both sides, the problem decouples into two independent problems). For a reflecting boundary, the boundary condition $u'(0) = 0$ selects only the even-numbered Hermite polynomials, $n = 0, 2, 4, \ldots$, and

$$\begin{aligned} Q(\mathbf{x},t) &\sim \exp\left(-at\sum_{i=0}^{N-1} 2i\right) \\ &= \exp[-atN(N-1)] \quad \text{(reflecting wall)}. \end{aligned} \tag{35}$$

For an absorbing boundary, the boundary condition $u(0) = 0$ selects the odd-numbered Hermite polynomials to give

$$Q(\mathbf{x}, t) \sim \exp\left(-at \sum_{i=1}^{N} (2i - 1)\right)$$
$$= \exp[-atN^2] \quad \text{(absorbing wall).} \tag{36}$$

In the following section we show how these results can be used to compute the survival probability of N vicious walkers in *zero* potential, with and without an absorbing or reflecting wall, by mapping the problem back to the oscillator problem.

4. Vicious walkers on a line

Here the case of N vicious walkers restricted by no potential is investigated. This problem can be solved in a quite simple way by mapping it to the problem of N vicious walkers in a harmonic potential and using the previous results. Again, we consider the Langevin equation (3), but with $V(\mathbf{x}) = 0$, and let all N vicious walkers start to move at time $t = t_0$. We introduce the following mapping from $\mathbf{x}$, t to the new coordinates $\mathbf{X}$, T by [7]

$$\mathbf{X} = \frac{\mathbf{x}}{\sqrt{2Dt}} \qquad t = t_0\, \mathrm{e}^{T}. \tag{37}$$

Then the Langevin equation (3) transforms to

$$\frac{\mathrm{d}X_i(T)}{\mathrm{d}T} = -\frac{1}{2} X_i(T) + \xi_i(T) \tag{38}$$

where $\xi_i(T) = \sqrt{t_0/2D}\, \mathrm{e}^{T/2} \eta_i(t_0\, \mathrm{e}^{T})$ is a Gaussian white noise with mean zero and correlator

$$\langle \xi_i(T) \xi_j(T') \rangle = \delta_{ij} \delta(T - T'). \tag{39}$$

The corresponding backward Fokker–Planck equation in the new coordinates is

$$\frac{\partial Q(\mathbf{X}, T)}{\partial T} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial X_i^2} Q(\mathbf{X}, T) - \frac{1}{2} \sum_{i=1}^{N} X_i \frac{\partial Q(\mathbf{X}, T)}{\partial X_i} \tag{40}$$

where the space coordinates are now the starting points of the vicious walkers, given by

$$X_i(T = 0) = \frac{x_i(t_0)}{\sqrt{2Dt_0}}. \tag{41}$$

In the new coordinates this problem looks identical to the harmonic potential problem with $a = 1/2$ and $D = 1/2$. Hence the asymptotic (in time) solution for the survival probability of N vicious walkers is, according to our previous results,

$$Q(\mathbf{X}, T) \sim \exp\left(-\frac{T}{2} \sum_{i=0}^{N-1} i\right) \det B^H \tag{42}$$

where $(B^H)_{nm} = H_{n-1}(X_m/\sqrt{2})$ and $n, m = 1, \ldots, N$. Mapping back to the original coordinates $(\mathbf{x}, t)$ leads to the survival probability

$$Q(\mathbf{x}, t) \sim \left(\frac{t}{t_0}\right)^{-\frac{1}{2}\sum_{i=0}^{N-1} i} \det B^L \tag{43}$$

where $(B^L)_{nm} = H_{n-1}(x_m/2\sqrt{Dt_0})$, with $n, m = 1, \ldots, N$. The determinant $\det B^L$ is proportional to the Vandermonde determinant [1]: $\det B^L = (Dt_0)^{-N(N-1)/2} \prod_{i<j} |x_i - x_j|$, and all t_0-dependence drops out, as it must, to give the long-time behaviour

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{4}} \tag{44}$$

which is just the result Fisher obtained [1]. But our approach to the problem also gives a simple way to obtain expressions for the survival probability for N vicious walkers with an absorbing or reflecting wall at the origin (and all walkers starting on one side of the wall).

The essential arguments have been given in the preceding subsection. For an absorbing (reflecting) boundary, only the odd (even) basis functions contribute. Note first that the Fisher result (44) follows immediately from (34) on setting $a = 1/2$ and $T = \ln(t/t_0)$. The detailed discussion above was given mainly to show how the arbitrary scale t_0 drops out. To obtain the asymptotic results for a reflecting or absorbing wall at the origin, we can simply make the same replacements in equations (35) and (36) respectively. For the absorbing boundary, we recover the result of Krattenthaler *et al* [3]:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N^2}{2}} \quad \text{(absorbing wall)} \tag{45}$$

while for a reflecting wall we obtain

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{2}} \quad \text{(reflecting wall).} \tag{46}$$

The latter is, to our knowledge, a new result.

As a final comment we note that the case where the absorbing or reflecting wall moves, with a displacement $x_w = ct^{1/2}$, is also amenable in principle to exact analysis. The change of variable (37) maps the problem to one where the N walkers move in a harmonic oscillator potential, and the absorbing or reflecting wall is at a *fixed position* in the new coordinates. This problem has been analysed for a single walker [8], and the survival probability decays as $t^{-\theta}$, where the exponent θ is found to vary continuously with the amplitude, c, of the wall displacement. The same qualitative features will be present for N vicious walkers. For a reflecting (R) or absorbing (A) boundary, one will obtain a decay exponent $\theta_{R,A} = N(N-1)/4 + f_{R,A}(c, N)$, where $f_{R,A}(-\infty, N) = 0$, corresponding to a rapidly receding wall, which will be equivalent to no wall at all, and $f_R(0, N) = N(N-1)/4$, $f_A(0, N) = N(N+1)/4$ correspond to a static wall.

5. Conclusion

In this paper we have derived the exact asymptotics for the survival probability of vicious walkers moving in a square-well potential and a harmonic potential with various combinations of absorbing and reflecting walls. The results for a harmonic potential have been used to find the properties of free vicious walkers (zero potential) through a change of variables, and a new result obtained for the case of a single reflecting boundary. Comparing all results for each potential one recognizes that the survival probability decays faster when the number of walls is increased, with absorbing walls causing a faster decrease than reflecting walls, in accord with intuitive expectations.

Note added in proof. After this paper was submitted for publication we learned that the result in equation (9) has been obtained independently, using a different method by M Katori and H Tanemura (*Preprint* math-ph/0402061).

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