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Drowsy cheetah hunting antelopes: a diffusing predator seeking fleeing prey

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Abstract. We consider a system of three random walkers (a 'cheetah' surrounded by two 'antelopes') diffusing in one dimension. The cheetah and the antelopes diffuse, but the antelopes experience in addition a deterministic relative drift velocity, away from the cheetah, proportional to their distance from the cheetah, such that they tend to move away from the cheetah with increasing time. Using the backward Fokker–Planck equation we calculate, as a function of their initial separations, the probability that the cheetah has caught neither antelope after infinite time.

Keywords: persistence (theory), stochastic particle dynamics (theory)

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1. Introduction

Diffusion controlled reactions of three particles in one dimension can be completely understood by mapping the process to a single diffusing particle in a two-dimensional wedge [1, 2], where the lines of reaction—the positions where two particles meet—correspond to the boundaries of the wedge. By this elegant method Fisher and Gelfand [1] investigated diffusing particles termed vicious walkers which annihilate on meeting, while Redner and Krapivsky [3, 4] studied the equivalent capture reaction, where a single diffusing prey ('lamb') is eliminated on meeting one of two diffusing predators ('lions') which start one on either side of the prey. One of the main properties of interest in these problems is the survival probability of all three vicious walkers or, equivalently, the single prey. The vicious walker and predator–prey processes are equivalent, as far as the survival probability is concerned, because the predators cannot meet each other without first meeting the prey. We note that the concept of vicious walkers has recently been extended to families of vicious walkers, in which members of one family only interact with members of other families [5].

In this paper we introduce a three-particle system in one dimension consisting of two prey ('antelopes'), surrounding a single predator ('cheetah'). So far this is just another statement of the vicious walker problem with three walkers. Our model differs from the standard model, however, as follows. Besides performing a diffusive motion all particles are subjected to a drift which increases linearly with their position coordinate. Considering the case where both species have the same diffusion constant, the equation of motion for the antelopes (A_1, A_2) and the cheetah (C) with initial positions $x_{A_1} < x_C < x_{A_2}$ is taken to be

$$\dot{x}_i = a x_i + \eta_i(t), \qquad i = A_1, A_2, C \tag{1}$$

where a is the strength of the drift. The Langevin noise $\eta_i(t)$ is a Gaussian white noise with mean zero and correlator

$$\langle \eta_i(t) \eta_j(t') \rangle = 2D\delta_{ij}\delta(t - t'). \tag{2}$$

Equation (1) models the overdamped motion of three particles moving independently in an inverted parabolic potential. The calculation of the time-dependent survival probability for three vicious walkers in a conventional parabolic potential (i.e. with $a < 0$ in equation (1)) has been presented elsewhere [6].

Studying the problem in the *relative* coordinates, $y_1 = x_C - x_{A_1}$ and $y_2 = x_{A_2} - x_C$, the equations of motion have terms linearly depending on these relative coordinates. Therefore the antelopes are always drifting away from the cheetah, with a drift rate proportional to the distance from the predator. As a result, there is a nonzero probability that both antelopes wander off to infinity without meeting the cheetah if they are initially separated from the cheetah. Defining the process to be 'alive' if neither of the antelopes has met the cheetah, we find a nonzero survival probability $Q(y_1, y_2)$ for $y_1, y_2 > 0$. The aim of this paper is to calculate this survival probability, $Q(y_1, y_2)$, in the limit of infinite time, given that the antelopes started initially at relative distances y_1 and y_2 from the cheetah.

To provide context for our result we consider first a cheetah and a single antelope. In section 3 the case of a cheetah surrounded by two antelopes is investigated by mapping the process to a single diffusing particle in a two-dimensional wedge. Section 4 is a short conclusion.

2. A cheetah and a single antelope

The dynamics of a cheetah (C) and an antelope ($A_1 = A$) is described by the Langevin equation (1) with noise correlator (2). The process terminates when the cheetah and the antelope meet, i.e. when $x_A = x_C$. Setting the initial positions as $x_A < x_C$, we introduce a relative coordinate $y_1 = y = x_C - x_A$ which obeys the Langevin equation:

$$\dot{y} = ay + \xi(t), \tag{3}$$

where $\xi(t) = \eta_C - \eta_A$ is a Gaussian white noise with mean zero and correlator

$$\langle \xi(t)\xi(t') \rangle = 4D\delta(t - t'). \tag{4}$$

The probability $Q(y)$ that the antelope has survived in the limit of infinite time, given that antelope and cheetah started at a relative distance y, satisfies the corresponding backward Fokker–Planck equation:

$$ay\frac{\mathrm{d}Q(y)}{\mathrm{d}y} + 2D\frac{\mathrm{d}^2Q(y)}{\mathrm{d}y^2} = 0. \tag{5}$$

Since the antelope is eliminated on meeting the cheetah, the survival probability has to vanish for $y = 0$: $Q(0) = 0$. If the prey is initially infinitely far from the predator it will certainly survive, so $Q(\infty) = 1$. Solving the backward Fokker–Planck equation (5) with the stated boundary conditions gives

$$Q(y) = \mathrm{Erf}\left(\sqrt{\frac{a}{4D}}y\right), \tag{6}$$

where $\mathrm{Erf}(x)$ is the error function. This result will occur again in the next section as a borderline case.

3. A cheetah surrounded by two antelopes

In this section we investigate the infinite-time survival probability of two antelopes surrounding a cheetah. To address the problem in a simple way, we interpret the individual one-dimensional coordinates of the antelopes and the cheetah, x_{A_1}, x_C, x_{A_2}, as the coordinates of a single diffusing particle in three dimensions, which are projected down to the diffusion of a single particle in a two-dimensional absorbing wedge in the space of relative coordinates. The boundary conditions imposed by the elimination process of the antelopes on meeting the cheetah correspond to the boundaries of the absorbing wedge.

The antelopes and the cheetah evolve according to the Langevin equation (1) with noise correlator (2). Mapping this process onto a single diffusing particle in a two-dimensional wedge, we use the relative coordinates $y_1 = x_C - x_{A_1}$ and $y_2 = x_{A_2} - x_C$. This diffusing particle now obeys the following equation of motion:

$$\dot{y}_j = a y_j + \xi_j, \qquad j = 1, 2, \tag{7}$$

where ξ_j is the 'relative' Gaussian white noise defined by $\xi_1 = \eta_C - \eta_{A_1}$ and $\xi_2 = \eta_{A_2} - \eta_C$. The mean is zero as beforehand but the correlator now becomes

$$\langle \xi_i(t)\xi_j(t')\rangle = \begin{cases} 4D\delta(t-t') & \text{for } i = j, \\ -2D\delta(t-t') & \text{for } i \neq j. \end{cases}$$

Note that exactly the same equations for the relative coordinates are obtained if the individual coordinates obey the equations $\dot{x}_{A_1} = a(x_C - x_{A_1}) + \eta_{A_1}$, $\dot{x}_C = \eta_C$, $\dot{x}_{A_2} = a(x_{A_2} - x_C) + \eta_{A_2}$. In this representation, the cheetah is only diffusing (hence 'drowsy'), while the antelopes have both diffusive and deterministic ('flight') components to their motion.

To determine the infinite-time survival probability of the equivalent single diffusing particle in two dimensions we consider the time-independent backward Fokker–Planck equation in the initial coordinates y_1, y_2:

$$a\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)Q(y_1,y_2) + 2D\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1\partial y_2}\right)Q(y_1,y_2) = 0. \tag{8}$$

Since an antelope is eliminated on meeting the cheetah, the survival probability of the single random walker must vanish when $y_1 = 0$ or $y_2 = 0$, corresponding to the absorbing boundaries of a wedge with opening angle $\Theta = \pi/2$, in which the single random walker is diffusing; see figure 1. If both antelopes are infinitely far from the cheetah, the survival probability will be unity, hence $Q(\infty, y_2) = Q(y_1, \infty) = 1$.

In order to reduce equation (8) to a canonical form, a change of variables is required. The variables are first rendered dimensionless by the change of variables $\tilde{y}_i = y_i\sqrt{a/2D}$, $i = 1, 2$. Introducing the new variables u and v according to

$$\tilde{y}_1 = \frac{u + \sqrt{3}v}{2} \qquad \tilde{y}_2 = \frac{u - \sqrt{3}v}{2}, \tag{9}$$

transforms equation (8) to

$$\left[u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right]Q(u,v) = 0. \tag{10}$$

The absorbing boundaries in the new variables u and v are at $u = \pm\sqrt{3}v$. In the new variables, therefore, the wedge is symmetric about the u-axis and has an opening angle of $\Theta = \pi/3$—see figure 1. Because of the symmetry of the wedge, polar coordinates (r, φ) are appropriate. Hence the time-independent backward Fokker–Planck equation becomes

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \left(\frac{1}{r} + r\right)\frac{\partial}{\partial r}\right]Q(r,\varphi) = 0. \tag{11}$$

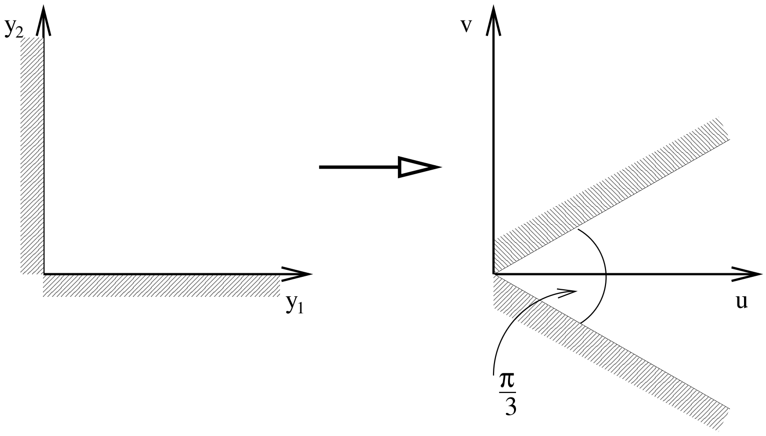

Figure 1. The transformation to a canonical differential equation maps the right-angled wedge in (y_1, y_2) coordinates to an axisymmetric wedge of opening angle $\Theta = \pi/3$.

The boundary conditions reduce to $Q(r, \pi/6) = Q(r, -\pi/6) = 0$ and $Q(r = 0, \varphi) = 0$ at the absorbing boundaries of the wedge and $Q(\infty, \varphi) = 1$ for $-\pi/6 < \varphi < \pi/6$ corresponding to the survival of both antelopes if they are initially at infinite distance from the cheetah.

The partial differential equation (11) can be solved by separation of variables,

$$Q(r, \varphi) = \sum_{n=1}^{\infty} A_n R_n(r) \Phi_n(\varphi), \tag{12}$$

where the angular part $\Phi_n(\varphi)$ is a cosine mode satisfying the angular boundary conditions,

$$\Phi_n(\varphi) = \cos(3(2n-1)\varphi), \tag{13}$$

and the coefficients A_n are to be determined by the radial boundary conditions.

Substituting the result for $\Phi_n(\varphi)$ in (11) yields the following ordinary differential equation for $R_n(r)$.

$$r^2 R_n''(r) + \left(r + r^3\right) R_n'(r) - 9(2n-1)^2 R_n(r) = 0. \tag{14}$$

By setting $r^2 = \zeta$ and $R_n(r) = \zeta^{3n-3/2} \rho_n(\zeta)$ this differential equation is transformed into

$$\zeta \rho_n''(\zeta) + \left(\frac{1}{2}\zeta + 6n - 2\right) \rho_n'(\zeta) + \frac{6n-3}{4} \rho_n(\zeta) = 0. \tag{15}$$

This ordinary differential equation is related to the confluent hypergeometric differential equation (see 2.273(9) in [7]). Defining $\zeta = 2\sigma$ and $\rho_n(\zeta) = \exp(-\sigma)\psi_n(\sigma)$, equation (15) reduces to the confluent hypergeometric differential equation, also called Kummer's equation [7, 8],

$$\sigma \psi_n''(\sigma) + (6n - 2 - \sigma)\psi_n'(\sigma) - (3n - \tfrac{1}{2})\psi_n(\sigma) = 0. \tag{16}$$

The solutions of this differential equation are known. The general solution can be written in terms of Kummer's functions of the first kind, $\mathrm{M}(a, b, z)$, and of the second kind, $\mathrm{U}(a, b, z)$, also denoted confluent hypergeometric functions of the first and second kind [8]:

$$\psi_n(\sigma) = B_n \mathrm{M}\left(3n - \tfrac{1}{2}, 6n - 2, \sigma\right) + C_n \mathrm{U}\left(3n - \tfrac{1}{2}, 6n - 2, \sigma\right), \tag{17}$$

where B_n and C_n are constants to be determined by the boundary condition. Note that we have introduced, for later convenience, a redundancy in the coefficients, having A_n, B_n and C_n when there are only two independent sets of coefficients. This redundancy will be removed below by an explicit choice of the coefficients B_n.

Substituting all former transformations, the result for $R_n(r)$ is

$$R_n(r) = B_n r^{6n-3} \mathrm{e}^{-r^2/2} \mathrm{M}\left(3n - \frac{1}{2}, 6n - 2, \frac{r^2}{2}\right) + C_n r^{6n-3} \mathrm{e}^{-r^2/2} \mathrm{U}\left(3n - \frac{1}{2}, 6n - 2, \frac{r^2}{2}\right). \tag{18}$$

The particular solution we are looking for has to vanish at $r = 0$ and approach a constant value for $r \to \infty$ to satisfy the boundary conditions. The confluent hypergeometric function of the first kind is unity when its argument is zero, $\mathrm{M}(a, b, 0) = 1$, whereas the hypergeometric function of the second kind, $\mathrm{U}(a, b, z)$, diverges as $z \to 0$ for $b > 1$ [8] which is the case in our solution, where $b = 6n - 2$, since $n > 0$. Hence we set $C_n = 0$ in the solution so that it vanishes at $r = 0$.

Now we investigate the behaviour of our solution in the limit $r \to \infty$. The asymptotic form of the hypergeometric function of the first kind for large arguments, $z \to +\infty$, is [8]

$$\mathrm{M}(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} \mathrm{e}^{z}. \tag{19}$$

Hence the radial solution approaches a constant value for $r \to \infty$.

$$\lim_{r \to \infty} R_n(r) = 2^{3n-3/2} \frac{\Gamma(6n-2)}{\Gamma(3n-1/2)} B_n. \tag{20}$$

To simplify the fitting to the boundary condition $Q(r = \infty, \varphi) = 1$ we eliminate the aforementioned redundancy in the expansion coefficients by choosing the constants B_n such that $R_n(\infty) = 1$ for all n, i.e. we choose $B_n = 2^{3/2-3n}(\Gamma(3n-1/2))/(\Gamma(6n-2))$. The coefficients A_n in equation (12) can be determined by imposing the boundary condition $Q(\infty, \phi) = 1$, i.e. $\sum_{n=1}^{\infty} A_n \cos[3(2n-1)\phi] = 1$, for ϕ in the interval $(-\pi/6, \pi/6)$. This gives

$$A_n = \frac{4}{\pi} \frac{(-1)^{n-1}}{2n-1}. \tag{21}$$

Finally, we simplify the radial solution by use of Kummer's formula [8]:

$$\mathrm{e}^{z} \mathrm{M}(a, b, -z) = \mathrm{M}(b - a, b, z). \tag{22}$$

Then the solution for the infinite-time survival probability of the single diffusing particle in a wedge becomes, in the dimensionless variables (r, φ),

$$Q(r, \varphi) = \sum_{n=1}^{\infty} 2^{-3n+7/2} \frac{\Gamma(3n-1/2)}{\pi(2n-1)\Gamma(6n-2)} (-1)^{n-1} \cos(3(2n-1)\varphi) \times r^{6n-3} \mathrm{M}\left(3n - \frac{3}{2}, 6n - 2, -\frac{r^2}{2}\right). \tag{23}$$

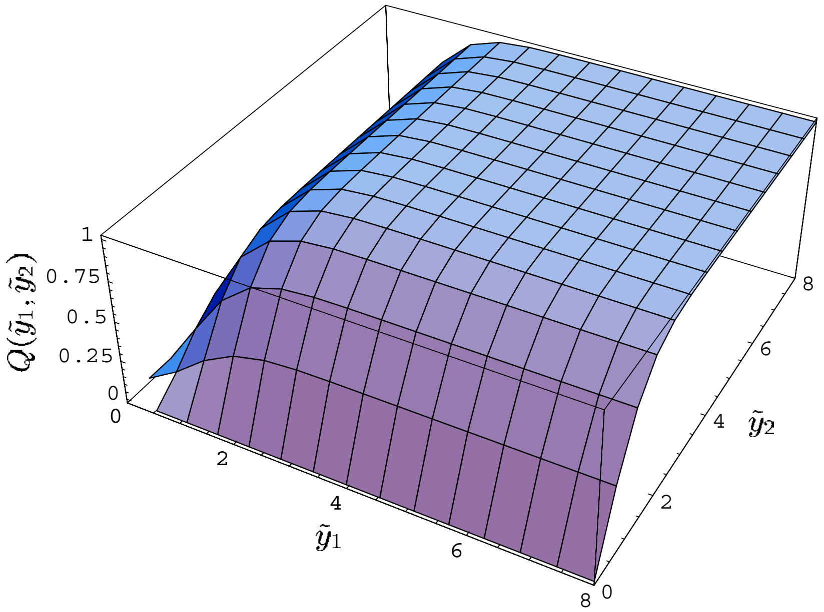

Figure 2. The infinite-time survival probability of two antelopes surrounding a cheetah, plotted against the dimensionless relative coordinates $\tilde{y}_1 = \sqrt{a/2D}(x_2 - x_1)$ and $\tilde{y}_2 = \sqrt{a/2D}(x_3 - x_2)$.

This sum is easily shown to converge since the summand a_n decays to zero faster than $1/n$ for $n \to \infty$. For large n, the confluent hypergeometric function approaches the exponential function, $\mathrm{M}\left(3n - \frac{3}{2}, 6n - 2, -(r^2/2)\right) \to \exp(-r^2/4)$. The asymptotic form of the quotient of gamma functions is given by $\Gamma(3n - 1/2)/\Gamma(6n - 2) \sim 2^{-3n+1}(6n - 3)^{-3n+3/2}\mathrm{e}^{3n-3/2}$. In summary, the summand decays to zero for large n as

$$a_n \sim \frac{2^{-6n+9/2}}{(2n-1)\pi}(6n-3)^{-3n+3/2} r^{6n-3} \mathrm{e}^{3n-3/2-r^2/4}, \tag{24}$$

where the alternating signs and oscillating cosine functions have been omitted. Although the sum clearly converges, the computational equipment was not sufficient to calculate the sum in general. Therefore, all plots of the solution to be displayed in this paper are approximations including the first 30 terms of the sum, which is sufficient in the chosen range, since, for example, the error due to the absence of the next ten terms, up to term 40, is smaller than 5×10^{-37}.

To plot and analyse the infinite-time survival probability we transform the solution back to the dimensionless relative coordinates $\tilde{y}_1$ and $\tilde{y}_2$. In those coordinates the result reads

$$\begin{aligned} Q(\tilde{y}_1, \tilde{y}_2) = \sum_{n=1}^{\infty} (-1)^{n-1} 2^{3n+1/2} \frac{\Gamma(3n - 1/2)}{\pi(2n-1)\Gamma(6n-2)} \cos\left[3(2n-1)\arctan\left(\frac{\tilde{y}_1 - \tilde{y}_2}{\sqrt{3}(\tilde{y}_1 + \tilde{y}_2)}\right)\right] \\ \times \mathrm{M}\left(3n - \tfrac{3}{2}, 6n-2, -\tfrac{2}{3}(\tilde{y}_1^2 + \tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right) \left(\tfrac{1}{3}(\tilde{y}_1^2 + \tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right)^{3n-3/2}. \end{aligned} \tag{25}$$

In figure 2 this function is plotted in the range $\tilde{y}_1, \tilde{y}_2 \in [0, 8]$. The survival probability smoothly increases from zero on the lines $\tilde{y}_1 = 0$ and $\tilde{y}_2 = 0$ to form a plateau of almost

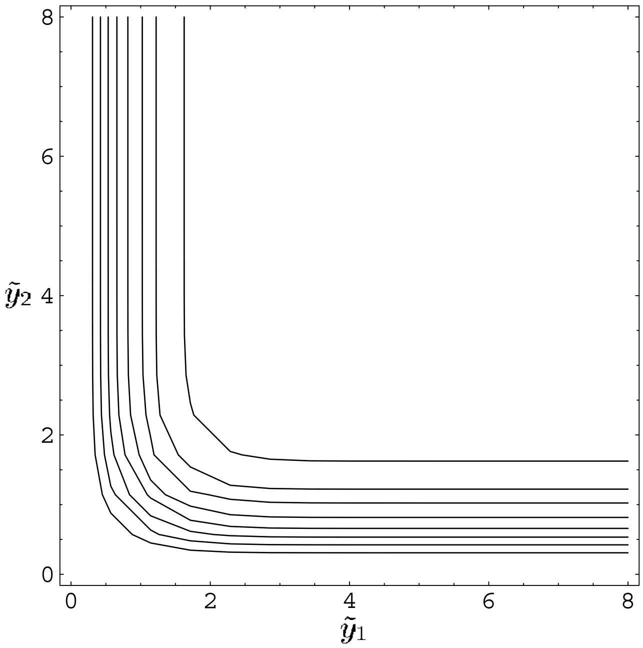

Figure 3. Contour lines of the infinite-time survival probability of two antelopes surrounding a cheetah versus the relative coordinates $\tilde{y}_1 = \sqrt{a/2D}(x_2 - x_1)$ and $\tilde{y}_2 = \sqrt{a/2D}(x_3 - x_2)$. The different lines correspond to constant probabilities of 0.1 up to 0.8.

constant probability for $\tilde{y}_1 > 2$ and $\tilde{y}_2 > 2$ that increases to unity at $\tilde{y}_1 = \infty$ and $\tilde{y}_2 = \infty$, corresponding to certain survival when both antelopes start infinitely far from the cheetah. Unfortunately Mathematica could not calculate the sum for $\tilde{y}_1 \to 0$ and $\tilde{y}_2 \to 0$, but the summand of equation (25) clearly vanishes when $\tilde{y}_1 = 0$ or $\tilde{y}_2 = 0$ due to the vanishing of the cosine functions. In particular, is is easy to show that for, say, $\tilde{y}_1 \to 0$ at fixed $\tilde{y}_2$, $Q(\tilde{y}_1, \tilde{y}_2)$ vanishes linearly with $\tilde{y}_1$, as is evident in figure 2.

Further analytical simplification can be made when both $\tilde{y}_1$ and $\tilde{y}_2$ tend to zero. In this case the $n = 1$ term dominates the sum in equation (25), and the argument z of the function $\mathrm{M(a, b, z)}$ can be set to zero. One then obtains, after some algebra, the simple form

$$Q(\tilde{y}_1, \tilde{y}_2) \to \frac{1}{\sqrt{2\pi}} \tilde{y}_1 \tilde{y}_2 (\tilde{y}_1 + \tilde{y}_2). \tag{26}$$

This function vanishes linearly with $\tilde{y}_1$ at fixed $\tilde{y}_2$, but as $\tilde{y}_1^3$ when $\tilde{y}_1$ is taken to zero with the ratio $\tilde{y}_2/\tilde{y}_1$ held fixed. Evidence of this behaviour can be observed near the origin in figure 2.

To study the survival probability further, it is also of interest to consider the contour lines of figure 2 as shown in figure 3. Investigating those one easily recognizes that the function is symmetric about the line $\tilde{y}_1 = \tilde{y}_2$, as it must be. Furthermore, in the limit of one relative coordinate tending to infinity, say $\tilde{y}_2 = \infty$, the problem with two antelopes simplifies to the problem of a single antelope with a cheetah, which has been calculated in section 2. In the dimensionless variables, the result for the survival probability of a single

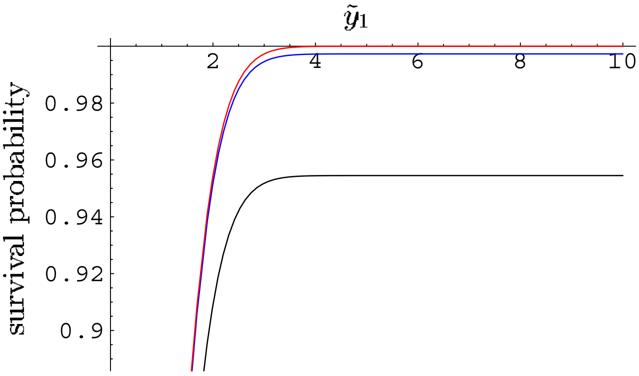

Figure 4. The infinite-time survival probability of two antelopes surrounding a cheetah keeping the relative coordinate $\tilde{y}_2 = c$ fixed at (bottom to top) $c = 2$, $c = 3$ and $c = 4$, where the top curve is already indistinguishable from the error function (27).

antelope and a cheetah is

$$Q(\tilde{y}_1, \infty) = \mathrm{Erf}\left(\frac{\tilde{y}_1}{\sqrt{2}}\right). \tag{27}$$

Unfortunately, extracting this limiting behaviour analytically has proved to be intractable. Instead, we plot $Q(\tilde{y}_1, \tilde{y}_2 = c)$ for $c = 2, 3, 4$; see figure 4. The figure clearly shows how the sequence of curves approaches the error function expected for $\tilde{y}_2 = \infty$; see equation (27). The $c = 4$ curve lies on top of the error function, demonstrating the limiting behaviour.

4. Conclusion

In this paper we introduced the interesting problem of a diffusion controlled reaction where, in addition to the diffusive motion, the particles are subjected to a separating drift. By mapping the process of two antelopes surrounding a cheetah to that of a single diffusing particle in two dimensions, we derived the probability that both antelopes have survived up to infinite time as a function of their initial separations from the cheetah.

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