VICIOUS WALKERS IN ONE-BODY POTENTIALS

THE UNIVERSITY of MANCHESTER

A thesis submitted to the University of Manchester for the degree of Master of Science in the Faculty of Science and Engineering

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Contents

List of Figures

Abstract

UNIVERSITY OF MANCHESTER

ABSTRACT OF THESIS submitted by **Karen Winkler** for the Degree of Master of Science and entitled **Vicious Walkers in One-body Potentials**

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This thesis is concerned with the survival probability of one-dimensional vicious walkers moving in one-body potentials. In particular the backward Fokker-Planck equation is used to derive the asymptotic form of the survival probability for large times.

A method is introduced explaining how the survival probability of a single random walker moving in an attractive one-body potential can be antisymmetrised to obtain the general survival probability of N vicious walkers moving therein. Using this approach, exact results are derived for vicious walkers in a square-well potential with absorbing or reflecting boundary conditions at the walls, and for a harmonic potential with an absorbing or reflecting boundary at the origin. In addition, by mapping the problem of vicious walkers in zero potential onto the harmonic potential, the survival probability of N vicious walkers on a line with or without an absorbing or reflecting wall at the origin is calculated.

Vicious walkers in an inverted harmonic potential are investigated in the case of N=3 by mapping the process onto a single random walker in a two-dimensional absorbing wedge. By this means the survival probability at infinite time is calculated.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Autobiographical Note

I started studying physics at the Universität Karlsruhe (TH) in October 2000, where I received my Vordiplom in August 2002. I continued my education at the Ludwig-Maximilians-Universität München until September 2003, when I interrupted my studies in Germany in order to take part in the Master of Science course at the University of Manchester.

Publications

The work described in Chapter 2 has been published in:

- "Vicious walkers in a potential",
  A. J. Bray and K. Winkler, *J. Phys. A: Math. Gen.*, **37**, 21 (2004) [1]

Chapter 1

Introduction

The random walker describing diffusion and Brownian motion is the best known model of a non-equilibrium stochastic process. Although investigated since the early twentieth century, the random walker model and its various extensions are still part of today's research and account for numerous applications in physics and chemistry.

This thesis is concerned with one-dimensional random walkers exhibiting contact interactions introduced as vicious walkers by M.E. Fisher in 1984 [2]. This model describes N walkers on a line taking steps at random to the left or to the right with equal probability. On meeting those random walkers are vicious and eliminate each other but do not interact otherwise.

Naturally arising from the definition of the vicious walkers model is the question of the survival probability of all N walkers, i.e. the probability that none of the N vicious walkers has met another up to a time t, see figure 1.1. This question becomes even more interesting if the vicious walkers are confined within a potential and subjected to absorbing or reflecting walls which eliminate the walkers or reverse their direction on contact. In most cases our main interest lies in the asymptotic decay in time of the survival probability. To this purpose it is convenient to work in continuous time and space as will be done throughout this thesis.

The remainder of this chapter will introduce the stochastic methods used to calculate

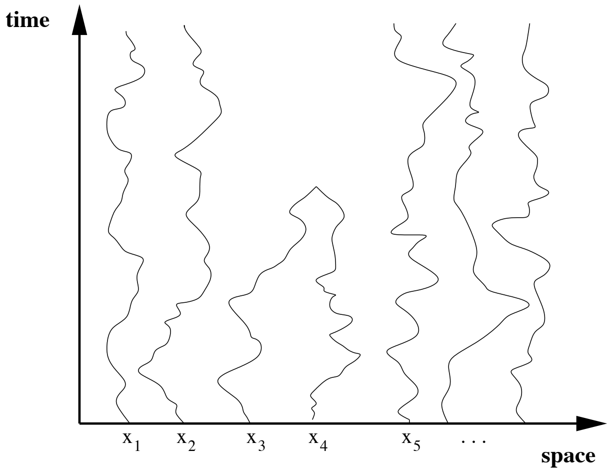

Figure 1.1: N vicious walkers on a line

the survival probabilities of random walkers. After a short section about the random walker, the Langevin equation will be presented followed by a derivation of the Fokker-Planck equation or rather the *backward* Fokker-Planck equation. In addition, the boundary conditions for absorbing and reflecting walls will be given. To familiarise the reader with the use of the backward Fokker-Planck equation, some applications will be examined at the end of this chapter.

In chapter 2, vicious walkers in different attractive one-body potentials will be investigated. At first, it will be explained how simple results for random walkers can be extended to give the survival probability of vicious walkers. In the proceeding, this method will be applied to vicious walkers in a square well and a harmonic potential, also giving access to vicious walkers in zero potential with or without a wall at the origin.

In the third chapter, diffusion processes in a two-dimensional wedge will be considered, giving means to calculate the survival probability of three vicious walkers in an inverted harmonic potential. At first a review of former work on diffusion processes in a wedge will be given. To this purpose the survival probability of a single random walker in a wedge with absorbing boundaries will be calculated. Afterwards a mapping will be introduced which brings problems of three vicious walkers in correspondence with a single walker in an absorbing wedge. To explain this mapping the calculation of the asymptotic survival probability of three vicious walkers on a line with different diffusion constants will be presented. Using this mapping the survival probability of three vicious walkers in an inverted harmonic potential will be calculated and visualised in the limit of infinite time.

In the last chapter a conclusion of all former results will be presented.

1.1 The random walker

Traditionally the random walker model is introduced on a equispaced lattice. For this purpose we consider a symmetric walker taking random steps of length l with equal probability to the left or to the right, where we define a the transition probability per unit time. Thus the walker arrives on sites $x = nl$ of the lattice, with n integral. After each step it loses every memory of its former position. Hence the probability $p(nl, t)$ of the random walker to be at one lattice position $x = nl$ at time t depends only on the probability that it occupied the nearest neighbour positions $(n+1)l$ or $(n-1)l$ or the position nl itself before. This allows the rate of change of the probability per unit time $\frac{dp(nl,t)}{dt}$ for a lattice point nl at time t to be written in the intuitive picture of the Master equation [3]:

$$\frac{dp(nl,t)}{dt} = a\,p(nl+l,t) + a\,p(nl-l,t) - 2a\,p(nl,t). \tag{1.1}$$

The rate of probability is increased by transitions from positions $(n+1)l$ and $(n-1)l$ and decreased by transitions out of lattice point nl either to the left or the right, see figure 1.2. The master equation can be solved by the method of generating function.

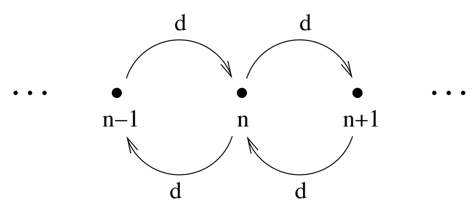

Figure 1.2: The rate of probability of a random walker to be at lattice position nl is increased by transitions from positions $(n+1)l$ and $(n-1)l$ and decreased by transitions out of lattice point nl either to the left or to the right.

For the purpose of this thesis we are more interested in the random walk in continuous variables x and t. Considering very small lattice spacings $l \ll 1$ both probabilities $p(nl+l,t)$ and $p(nl-l,t)$ can be expanded in a Taylor series in the small parameter l, keeping $x = nl$:

$$\begin{aligned}\frac{dp(x,t)}{dt} &= a\left(p(x,t) + \frac{\partial p(x,t)}{\partial x}l + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}l^2\right) \\ &+ a\left(p(x,t) - \frac{\partial p(x,t)}{\partial x}l + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}l^2\right) - 2a\,p(x,t) + \mathcal{O}(l^2). \quad (1.2)\end{aligned}$$

Taking the limit $l \rightarrow 0$, x becomes continuous and equation (1.2) turns into the diffusion equation, where we identify $al^2 = D$ as the diffusion constant:

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}. \quad (1.3)$$

Provided the random walker started at time $t = 0$ at position $x = x_0$ the solution of the diffusion equation is given by a normalised Gaussian function centred around x_0 with standard deviation $\sigma = \sqrt{\langle (x-x_0)^2\rangle} = \sqrt{2Dt}$:

$$p(x,t|x_0,0) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(x-x_0)^2}{4Dt}}. \quad (1.4)$$

Hence the probability density of the random walker is a $\delta-$function with unit magnitude at time zero and becomes a Gaussian function for times greater than zero. This Gaussian function is highly peaked for small times and spreads out as a square root of time for large times, see figure 1.3.

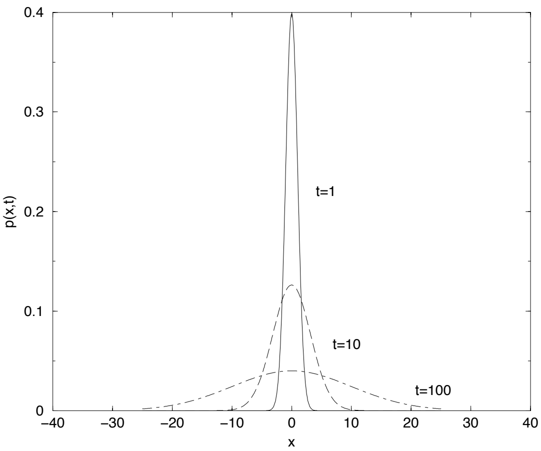

Figure 1.3: The Gaussian distribution of probability for a random walker starting at time zero at $x_0 = 0$ for three different times.

1.2 A stochastic equation of motion: the Langevin equation

The Langevin equation was originally introduced to explain Brownian motion, first observed for pollen grains. Classically Newton's equation of motion for a pollen grain in a fluid would include only a friction force $m\dot{\mathbf{v}} = -\alpha\mathbf{v}$ due to the loss of momentum of the particle during collisions. Thus it would describe a decelerating particle, which is in contrast to the equipartition theorem of statistical mechanics. This states that the mean kinetic energy of a particle is always greater than zero for temperatures greater than zero:

$$\frac{1}{2}m\left\langle \mathbf{v}^2 \right\rangle = \frac{3}{2}kT, \tag{1.5}$$

where T is the temperature in Kelvin and k is Boltzmann's constant. To account for this fact, Langevin added to Newton's equation of motion for a pollen grain a rapidly fluctuating random force $\mathbf{F}(t)$ [4]:

$$m\dot{\mathbf{v}} = -\alpha\mathbf{v} + \mathbf{F}(t). \tag{1.6}$$

The stochastic force $\mathbf{F}(t)$ is very irregular and its average over the ensemble is zero:

$$\langle \mathbf{F}(t) \rangle = 0. \tag{1.7}$$

Furthermore, the random force is assumed to be totally uncorrelated, i.e. the average value of the product of two random forces at different times t and t' or different components is zero:

$$\langle F_i(t)F_j(t') \rangle = q\delta_{ij}\delta(t - t'), \tag{1.8}$$

where i and j indicate components of the vector force $\mathbf{F}(t)$. For a measurement time scale much larger than the microscopic time scale of the collisions, this assumption of independence is a good approximation to reality. In terms of probability theory, the differential equation and the force correlator correspond to the Markov assumption, which states that the conditional probability of a particle moving with velocity $\mathbf{v}$ at time t is fully determined by its initial velocity $\mathbf{v}_0$ at time t_0. This characteristic is exhibited by the Langevin equation since it is a first order differential equation, i.e.

only one set of initial conditions is necessary and since the random force is δ-correlated, forces at times t' and t are independent [5].

From section 1.1 we already know the mean square displacement of a diffusion process to be $\langle (x - x_0)^2 \rangle = 2Dt$. For comparison it is also of interest to calculate the mean square displacement of a Brownian particle. Solving equation (1.6) for the one-dimensional case, for example by the method of integrating factor, yields:

$$v(t) = v_0 e^{-\frac{\alpha}{m}t} + \frac{1}{m} \int_0^t ds\, e^{-\frac{\alpha}{m}(t-s)} F(s), \tag{1.9}$$

where v_0 is the initial velocity at time zero. Integrating this result over time, we get the position of the particle dependent on time $x(t)$:

$$x(t) = x_0 + \frac{m}{\alpha} v_0 \left(1 - e^{-\frac{\alpha}{m}t}\right) + \frac{1}{\alpha} \int_0^t ds \left(1 - e^{-\frac{\alpha}{m}(t-s)}\right) F(s). \tag{1.10}$$

To obtain the second moments we take the squares of (1.9) and (1.10), respectively. Evaluating the mean and correlator of the random force $F(t)$ according to (1.7) and (1.8) results in

$$\left\langle v^2(t) \right\rangle = \left(v_0^2 - \frac{q}{2m\alpha} \right) e^{-\frac{\alpha}{m}2t} + \frac{q}{2m\alpha} \tag{1.11}$$

and

$$\left\langle (x(t) - x_0)^2 \right\rangle = \frac{m^2}{\alpha^2} \left(v_0^2 - \frac{q}{2m\alpha} \right) \left(1 - e^{-\frac{\alpha}{m}t}\right)^2 + \frac{q}{\alpha^2} \left[t - \frac{m}{\alpha} \left(1 - e^{-\frac{\alpha}{m}t}\right) \right]. \tag{1.12}$$

Hence in the stationary state, i.e. for infinite time, the second moment of the velocity becomes:

$$\left\langle v^2(t) \right\rangle = \frac{q}{2m\alpha}. \tag{1.13}$$

Taking such a stationary state distribution as initial velocity v_0 the first term in equation (1.12) vanishes and the mean square displacement behaves to leading order in time as

$$\left\langle (x(t) - x_0)^2 \right\rangle = \frac{q}{\alpha^2} t, \tag{1.14}$$

in correspondence to the random walker, where we identify $D = q/2\alpha^2$. Comparing the mean square velocity in equation (1.13) with the equipartition theorem in one dimension, we see that $q/2\alpha = kT$. Summarising, the diffusion constant of Brownian motion becomes:

$$D = \frac{kT}{\alpha}. \tag{1.15}$$

This is just the famous Einstein relation [6], which Einstein derived by different means before Langevin wrote down his equation.

To obtain a first order stochastic differential equation depending on the position of the pollen grain $\mathbf{x}$ the strong damping limit is taken in the original Langevin equation (1.6). Neglecting the inertial term leads to:

$$\alpha\dot{\mathbf{x}} = \mathbf{F}(t).$$

This equation now describes a stochastic variable, which responds instantaneously to a random force, which is the case of Brownian motion to a good approximation and corresponds to the movement of a random walker. We define a Langevin noise $\eta_i(t) = F_i(t)/\alpha$ with correlator

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D_{ij}\delta(t-t'), \tag{1.16}$$

where D_{ij} are the entries of a general matrix $\mathbf{D}$. In the case of a single random walker the different components of the stochastic force are not correlated and the matrix $\mathbf{D}$ is proportional to the identity matrix $D_{ij} = D\delta_{ij}$. Hence the correlator for a random walker is:

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t'). \tag{1.17}$$

Note that by this definition the mean square displacement of a random walker is just $\langle (x-x_0)^2\rangle = 2Dt$ as we found in section 1.1.

In the following we are interested in N one-dimensional walkers with position coordinates $(x_1, x_2, \ldots, x_N) = \mathbf{x}$ moving in a one-body potential, i.e. each walker is in the same individual potential. To this purpose conservative forces of the form $F_i(\mathbf{x}) = -\alpha\frac{\partial}{\partial x_i}V(\mathbf{x})$ are included, where the index i refers to the force acting on the i^{th} walker. Hence the equation of motion for the walker i becomes:

$$\dot{x}_i = -\frac{\partial V}{\partial x_i} + \eta_i(t), \tag{1.18}$$

where the Langevin noise is a Gaussian white noise with zero mean and correlator as defined above (1.17). This Langevin equation again obeys the Markov assumption.

Of particular importance in later chapters are conservative forces of the form $F_i(\mathbf{x}) = -\sum_{i=1}^{N} \gamma_{ij} x_j$. Stochastic processes which evolve according to such linear Langevin equation are called Ornstein-Uhlenbeck processes [7, 8] and for vanishing matrix ($\gamma_{ij} = 0$) they reduce to a so-called Wiener process [9].

1.3 The Fokker-Planck equations

Instead of working with differential equations for the stochastic variables x_i, $i = 1, 2, \ldots, N$ one may examine the same process as the evolution of the probability of all particles being at position $\mathbf{x}$ at time t given that they were at initial position $\mathbf{x}'$ at time t'. The forward Fokker-Planck equation [10, 11], also called forward Kolmogorov equation [12] or Smoluchowski equation, describes this development of the conditional probability $p(\mathbf{x}, t|\mathbf{x}', t')$. It can be derived from the Langevin equations (1.18) and (1.16):

$$\frac{\partial p(\mathbf{x}, t|\mathbf{x}', t')}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[\frac{\partial V(\mathbf{x})}{\partial x_i} p(\mathbf{x}, t|\mathbf{x}', t') \right] + \sum_{i,j=1}^{N} D_{ij} \frac{\partial^2}{\partial x_i \partial x_j} p(\mathbf{x}, t|\mathbf{x}', t'), \quad (1.19)$$

where the first term on the right hand side corresponds to drift and the second to diffusion of probability. The initial condition is given by:

$$p(\mathbf{x}, t = t'|\mathbf{x}', t') = \delta(\mathbf{x} - \mathbf{x}'). \quad (1.20)$$

For calculating first passage processes such as survival probabilities the second Fokker-Planck equation called *backward* Fokker-Planck equation is more appropriate. It is termed backward because this differential equation describes the evolution of probability with respect to the initial variables $\mathbf{x}', t'$. In order to derive the backward equation we consider the Langevin equation stated in equation (1.18) with Gaussian white noise (1.16). Integrating the Langevin equation over time from $t = t'$ to $t = t' + \Delta t$ with $x_i' = x_i(t')$ yields [13]:

$$x_i(t' + \Delta t) = x_i' - \frac{\partial V(\mathbf{x}')}{\partial x_i'} \Delta t + \eta_{i_{\Delta t}} + \mathcal{O}(\Delta t)^2. \quad (1.21)$$

where $\eta_{i_{\Delta t}}$ denotes the integral over time of the Langevin noise, which has zero mean and correlator $\left\langle \eta_{i_{\Delta t}} \eta_{j_{\tilde{\Delta t}}} \right\rangle = 2D_{ij}\mathrm{min}[\Delta \mathrm{t}, \tilde{\Delta \mathrm{t}}]$ in correspondence to the Gaussian white noise.

The conditional probability $p(\mathbf{x}, t|\mathbf{x}', t')$ of the random walker being at position $\mathbf{x}$ at time t given that it started at $(\mathbf{x}', t')$ is equal to the conditional probability $p(\mathbf{x}, t|\mathbf{x}(t'+\Delta t), t'+\Delta t)$ that the random walker started at a slightly later time $t'+\Delta t$ at a slightly later position $\mathbf{x}(t'+\Delta t)$ averaged over the noise $\eta_{\Delta t}$, see figure 2.10.

$$p(\mathbf{x}, t|\mathbf{x}', t') = \langle p(\mathbf{x}, t|\mathbf{x}(t'+\Delta t), t'+\Delta t)\rangle_{\eta_{\Delta t}}$$

By averaging over the noise we take into account all different positions $\mathbf{x}(t'+\Delta t)$ at time $t+\Delta t$, which the walker could occupy obeying the Langevin equation. Therefore all possible ways from $(\mathbf{x}', t')$ to $(\mathbf{x}, t)$ are included.

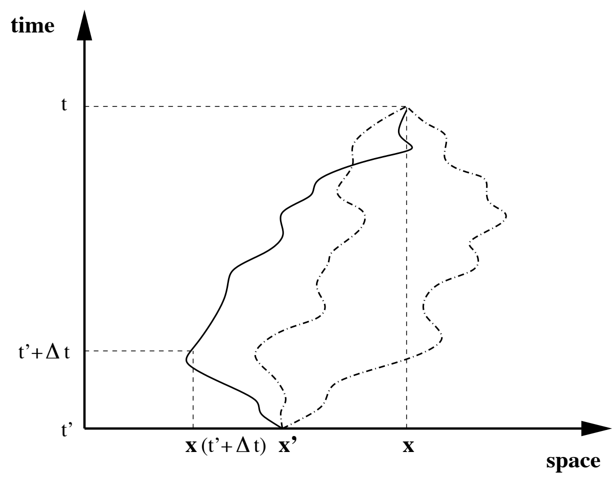

Figure 1.4: The black line shows a way a random walker could take from $(\mathbf{x}', t')$ to $(\mathbf{x}, t)$, being at a particular position $\mathbf{x}(t'+\Delta t)$ at time $t'+\Delta t$. But various other ways are possible, represented by the dash-dotted lines, being at different positions at time $t'+\Delta t$. These ways are taken into account by averaging over the noise.

The position at the later initial time $t'+\Delta t$ is just given by the integrated Langevin equation (1.21).

$$p(\mathbf{x}, t|\mathbf{x}', t') = \langle p(\mathbf{x}, t|\mathbf{x}' - \nabla' V(\mathbf{x}')\Delta t + \eta_{\Delta t}, t'+\Delta t)\rangle_{\eta_{\Delta t}} + \mathcal{O}(\Delta t)^2$$

Expanding the right hand side up to order $(\Delta t)^2$ leads to:

$$p_{t'} = \left\langle p_{t'} + \sum_{i=1}^{N}\left[-\frac{\partial V(\mathbf{x}')}{\partial x_i'}\Delta t + \eta_{i_{\Delta t}}\right]\frac{\partial p_{t'}}{\partial x_i'} + \frac{1}{2}\sum_{i,j=1}^{N}\eta_{i_{\Delta t}}\eta_{j_{\Delta t}}\frac{\partial^2 p_{t'}}{\partial x_i'\partial x_j'} + \frac{\partial p_{t'}}{\partial t'}\Delta t\right\rangle_{\eta_{\Delta t}},$$

where $p_{t'}$ is a shorthand for $p(\mathbf{x}, t|\mathbf{x}', t')$. Evaluating the average by using the mean and correlator of the integral over time of the Langevin noise yields:

$$p_{t'} = p_{t'} - \sum_{i=1}^{N} \frac{\partial V(\mathbf{x}')}{\partial x_i'} \frac{\partial p_{t'}}{\partial x_i'} \Delta t + \sum_{i,j=1}^{N} D_{ij} \frac{\partial p_{t'}}{\partial x_i' \partial x_j'} \Delta t + \frac{\partial p_{t'}}{\partial t'} \Delta t.$$

Subtracting $p_{t'}$ from both sides and dividing by Δt in the last equation results in the desired backward Fokker-Planck equation:

$$-\frac{\partial p(\mathbf{x}, t|\mathbf{x}', t')}{\partial t'} = -\sum_{i=1}^{N} \frac{\partial V(\mathbf{x}')}{\partial x_i'} \frac{\partial p(\mathbf{x}, t|\mathbf{x}', t')}{\partial x_i'} + \sum_{i,j=1}^{N} D_{ij} \frac{\partial^2 p(\mathbf{x}, t|\mathbf{x}', t')}{\partial x_i' \partial x_j'}. \tag{1.22}$$

Similar to the forward Fokker-Planck equation the backward Fokker-Planck equation can be separated into a diffusion term consisting of the second order derivative term and a drift term represented by the first order derivative term, which is proportional to the derivative of the potential. Hence any potential causes the walkers to be biased into a direction depending on their position. Despite this similarity there is an essential difference between the forward and backward Fokker-Planck equation. The forward equation describes the evolution of the probability for $t > t'$ with $\mathbf{x}', t'$ kept fixed whereas in the backward equation the final values $\mathbf{x}, t$ maintain constant and the development in $t' < t$ is observed leading to the final state. This is the reason why the backward Fokker-Planck equation is an important tool for first passage processes such as survival probabilities.

Regarding random walkers in an interval with absorbing boundaries at one or both ends, the survival probability $Q(\mathbf{x}, t)$ may be interpreted as the sum over the probabilities of all final states, at which the walkers are termed to be alive, hence those inside the interval.

$$Q(\mathbf{x}, t) = \int_{interval} d\mathbf{y}\, p(\mathbf{y}, t|\mathbf{x}, 0) = \int_{interval} d\mathbf{y}\, p(\mathbf{y}, 0|\mathbf{x}, -t), \tag{1.23}$$

where the last equality is valid since the system is homogeneous in time, which is true as long as the potential is time independent as assumed throughout this thesis [9]. Clearly the survival probability only depends on the initial values $\mathbf{x}, t$ of the probability $p(\mathbf{y}, 0|\mathbf{x}, -t)$ and therefore obeys the backward Fokker-Planck equation as stated in equation (1.22) for $t' = -t$ and $\mathbf{x}' = \mathbf{x}$:

$$\frac{\partial Q(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial V(\mathbf{x})}{\partial x_i} \frac{\partial Q(\mathbf{x},t)}{\partial x_i} + \sum_{i,j=1}^{N} D_{ij} \frac{\partial^2 Q(\mathbf{x},t)}{\partial x_i \partial x_j}. \tag{1.24}$$

In general the survival probability can be defined as the probability that none of the N walkers has been eliminated up to time t, provided that they started at initial positions $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.

The initial condition is $Q(\mathbf{x}, t = 0) = 1$ for $\mathbf{x} \in$ *interval* and zero otherwise, which follows from the initial condition in equation (1.20). While examining the survival probability of problems, boundary conditions have to be taken into account, which will be introduced in the next section.

1.3.1 Boundary conditions for the *backward* Fokker-Planck equation

The two main types of boundary conditions used in this thesis are absorbing and reflecting boundaries. Consider a random walker initially confined within a region **R** with surface **S** [9].

absorbing boundary: For **S** an absorbing boundary the walker is eliminated on contact. Hence the probability of the walker being on the surface initially is zero: $p(\mathbf{y}, t|\mathbf{x}, 0) = 0$ for $\mathbf{x} \in \mathbf{S}$. From the definition of the survival probability in equation (1.23) it follows that also the survival probability is zero at the surface $Q(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \mathbf{S}$.

reflecting boundary: In the case of a reflecting boundary at the surface the random walker cannot penetrate the boundary and hence the rate of change of probability with respect to the *initial* coordinates is zero perpendicular to the boundary: $\sum_{i,j}^{N} n_i D_{ij} \frac{\partial p(\mathbf{y},t|\mathbf{x},0)}{\partial x_j} = 0$, where $\mathbf{n}$ is normal to the surface. For N one-dimensional random walkers this constraint reduces to $\frac{\partial p(\mathbf{y},t|\mathbf{x},0)}{\partial x_i} = 0 \; \forall i \in N$ and analogous for the survival probability $\frac{\partial Q(\mathbf{x},t)}{\partial x_i} = 0 \; \forall i \in N$.

1.4 Applications of the *backward* Fokker-Planck equation

To illustrate the use of the backward Fokker-Planck equation some applications are presented in this section also broadening the understanding of the random walker.

At first a random walker in a spherical domain in d dimensions is considered. The walker is subjected to movements between an inner and an outer shell. Shrinking the radius of the inner shell to zero and expanding the radius of the outer shell to infinity the probability of visiting the origin is investigated observing different behaviour for dimensions greater and smaller than two.

Afterwards the survival probability of a random walker in a harmonic potential with an absorbing boundary at the origin is calculated giving access to the survival probability of two vicious walkers in a harmonic potential.

1.4.1 The random walker in a spherical domain

In the interest of examining the probability of visiting the origin $Q_0(r)$ for a random walker in d dimensions we consider at first a random walker between two concentric spheres, see figure 1.5. For our purpose it is useful to calculate the first passage probability $Q(r)$ that the random walker hits the inner sphere without having hit the outer one. In the limit of zero radius for the inner sphere r_{in} and infinite radius for the outer sphere r_{out} the first passage probability $Q(r)$ corresponds to the probability of visiting the origin $Q_0(r)$.

The first passage probability $Q(r)$ of a random walker between two d dimensional spheres can be interpreted as the survival probability of the random walker starting at radius r, $r_{\text{in}} < r < r_{\text{out}}$, if we define the outer sphere to be an absorbing boundary, hence $Q(r = r_{\text{out}}) = 0$, and the inner sphere to be a trap, at which the random walker is kept alive, therefore $Q(r = r_{\text{in}}) = 1$. For our calculation it is sufficient to examine the survival probability in the infinite time limit, where the survival probability does not change with time anymore. For a single random walker in the infinite time limit subjected to no potential the backward Fokker-Planck equation (1.24) reduces to a

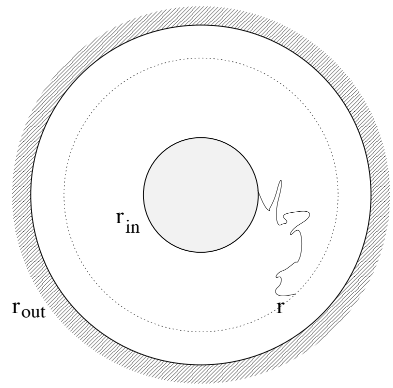

Figure 1.5: A random walker between two concentric spheres

simple Laplace equation:

$$\nabla^2 Q(r) = 0. \tag{1.25}$$

Because of the spherical symmetry only the radial part of the Laplacian is nonzero. In the general case of d dimensions the Laplacian becomes:

$$\frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\right)Q(r) = 0. \tag{1.26}$$

The solution to this differential equation is of the form $Q(r) = A + B\ln r$ for $d = 2$ and $Q(r) = A + B/r^{d-2}$ otherwise, where A and B are constants. Determining these constants by use of the above boundary conditions yields:

$$Q(r) = \begin{cases} \frac{\ln\frac{r_{\text{out}}}{r}}{\ln\frac{r_{\text{out}}}{r_{\text{in}}}} & \text{for } d = 2 \\ \frac{1-\left(\frac{r_{\text{out}}}{r}\right)^{d-2}}{1-\left(\frac{r_{\text{out}}}{r_{\text{in}}}\right)^{d-2}} & \text{for } d \neq 2 \end{cases}$$

Now we are able to calculate the limits of $r_{\text{in}} \to 0$ and $r_{\text{out}} \to \infty$ in order to get the probability $Q_0(r)$ that the random walker visits the origin before it wanders off to infinity provided it started at position r. We investigate the three different cases $d < 2$, $d = 2$ and $d > 2$ separately.

$d < 2$

At first we take the limit of $r_{\text{in}} \to 0$, giving:

$$\lim_{r_{\text{in}}\to 0} Q(r) = 1 - \left(\frac{r_{\text{out}}}{r}\right)^{d-2}. \tag{1.27}$$

Since $r_{\text{out}} > r$ by the setting of the problem, there is always a finite probability that the random walker visits the origin. Taking the outer sphere to infinity the probability to hit the origin before wandering off to infinity becomes $Q_0(r) = 1$. The same result is obtained if we evaluate the limit $r_{\text{out}} \to \infty$ first:

$$\lim_{r_{\text{out}} \to \infty} Q(r) = 1. \tag{1.28}$$

Hence in dimensions smaller than two the random walker will always visit the origin and by translational symmetry this implies, that the random walker will visit every point in space. This property of the random walk is called *recurrence* [14].

$d > 2$

In the case of dimensions greater than two evaluating the limit of zero radius for the inner sphere yields:

$$\lim_{r_{\text{in}} \to 0} Q(r) = \lim_{r_{\text{in}} \to 0} \left(\frac{r_{\text{in}}}{r_{\text{out}}}\right)^{d-2} - \left(\frac{r_{\text{in}}}{r}\right)^{d-2} = 0. \tag{1.29}$$

Hence independent of the radius of the outer sphere the probability of hitting the origin is zero $Q_0(r) = 0$. This result also implies that two random walker never meet in $d > 2$ dimensions and also in $d = 2$ dimensions as will be seen in the next section. Hence it is illogical to define point size vicious walkers in $d \geq 2$ dimensions, since their contact interaction is never applied.

Again the result does not change if the limit of $r_{\text{out}} \to \infty$ is taken before $r_{\text{in}} \to 0$:

$$\lim_{r_{\text{in}} \to 0} \lim_{r_{\text{out}} \to \infty} Q(r) = \lim_{r_{\text{in}} \to 0} \left(\frac{r_{\text{in}}}{r}\right)^{d-2} = 0. \tag{1.30}$$

In general the result states that in dimensions greater than two the random walker is only visiting a small amount of the total space unlikely to revisit any particular point, this property is called *transience* [15, 16].

$d = 2$

So far $Q(r)$ has been continuous throughout space, therefore interchanging the two limits for $d < 2$ and $d > 2$ did not make any difference. In the marginal case of two dimensions this is not true anymore. The two opposing results are:

$$\lim_{r_{\text{out}}\to\infty}\lim_{r_{\text{in}}\to 0} Q(r) = 0$$
$$\lim_{r_{\text{in}}\to 0}\lim_{r_{\text{out}}\to\infty} Q(r) = 1 \tag{1.31}$$

Hence the two dimensional random walker on the continuum is neither transient nor recurrent.

1.4.2 The random walker in a harmonic potential

In this section we investigate the one-dimensional random walker in a harmonic potential with an absorbing wall at the origin [17]. This example will also be of relevance as a borderline case for results derived in chapter 2 and 3.

Considering a harmonic potential $V(x) = \mu x^2/2$ the Langevin equation for the Ornstein-Uhlenbeck process of one random walker reads:

$$\dot{x} = -\mu x + \eta(t), \tag{1.32}$$

where the Langevin noise is the Gaussian white noise as introduced beforehand with zero mean and correlator $\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$. To calculate the survival probability of the random walker the corresponding one-dimensional backward Fokker-Planck equation is the appropriate equation to be solved.

$$\frac{\partial Q(x,t)}{\partial t} = -\mu x\frac{\partial Q(x,t)}{\partial x} + D\frac{\partial^2 Q(x,t)}{\partial x^2} \tag{1.33}$$

Provided that the random walker starts on the positive side away from the origin the initial condition for the survival probability is: $Q(x, t = 0) = 1$. Due to the absorbing boundary at the origin the solution of the differential equation must vanish at $x = 0$ ($Q(x = 0, t) = 0\ \forall t$). If the random walker is at an infinite distance from the origin, there is no possibility that it finally reaches the wall and gets eliminated in finite time, therefore the survival probability for the walker starting at infinity is set to one ($Q(x = \infty, t) = 1\ \forall t$). The backward Fokker-Planck equation with those initial and boundary conditions can be solved by mapping it to the diffusion equation by defining [18]

$$\tau = \frac{1}{2\mu}\left(1 - e^{-2\mu t}\right) \quad z = xe^{-\mu t}.$$

Hence the differential equation becomes

$$\frac{\partial Q(z,\tau)}{\partial \tau} = D\frac{\partial^2 Q(z,\tau)}{\partial z^2},$$

where the boundary conditions have been preserved, i.e. $Q(z=0,\tau)=0\ \forall\tau$ and $Q(z=\infty,\tau)=1\ \forall\tau$. The solution to the diffusion equation with the given boundary conditions is well known, expressed in the initial coordinates the solution is given by:

$$Q(x,t) = \mathrm{Erf}\left(\frac{e^{-\mu t}}{\sqrt{2D(1-e^{-2\mu t})}}\sqrt{\mu}x\right), \tag{1.34}$$

where $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x du\, e^{-u^2}$ is the error function. For large times the survival probability decays exponentially with time, $Q(x,t) \sim xe^{-\mu t}$.

This result also describes the survival probability of two one-dimensional vicious walkers in a harmonic potential. Each vicious walker obeys a Langevin equation of form (1.32) for coordinates x_1 and x_2, respectively, with $x_1 \leq x_2$. Introducing the relative coordinate $x_{12} = x_2 - x_1$ and 'relative noise' $\xi(t) = \eta_2 - \eta_1$ leads to the Langevin equation of a single random walker:

$$\dot{x}_{12} = -\mu x_{12} + \xi(t), \tag{1.35}$$

where now the noise has zero mean and correlator $\langle\xi(t)\xi(t')\rangle = 4D\delta(t-t')$. Hence substituting $2D$ instead of D in equation (1.33) gives the analogous backward Fokker-Planck equation. Since the vicious walkers annihilate on meeting, the survival probability has to be zero for $x_1 = x_2$, i.e. $x_{12} = 0$, whereas at infinite distance from each other the vicious walkers will definitely survive, hence $Q(x_{12}=\infty,t)=1$. These are just the same boundary conditions as in the case of a single random walker with an absorbing boundary at the origin. Therefore the survival probability of two vicious walkers in a harmonic potential is given by solution (1.34), with $2D$ instead of D:

$$Q(x,t) = \mathrm{Erf}\left(\frac{e^{-\mu t}}{\sqrt{4D(1-e^{-2\mu t})}}\sqrt{\mu}x\right), \tag{1.36}$$

and decays for large times as

$$Q(x_{12},t) \sim x_{12}e^{-\mu t}. \tag{1.37}$$

Chapter 2

Vicious walkers in an attractive potential

In this chapter we derive exact results for the asymptotic form of the survival probability $Q(\mathbf{x}, t)$ of N one-dimensional vicious walkers moving in an attractive one-body potential. Defining the variables $(x_1, \ldots, x_N) = \mathbf{x}$ as the initial positions of the vicious walkers we use the backward Fokker-Planck equation to compute explicit results for a square-well potential with absorbing or reflecting boundary conditions at the walls, and for a harmonic potential with an absorbing or reflecting boundary at the origin and the walkers starting on the positive half line. By mapping the problem of N vicious walkers in zero potential onto the harmonic potential problem, we derive known results for vicious walkers on an infinite line and on a semi-infinite line with an absorbing wall at the origin. This mapping also provides means to derive a new result for vicious walkers on a semi-infinite line with a reflecting boundary at the origin.

2.1 Introduction

On introducing the model of vicious walkers Fisher and Huse [2, 19] determined the survival probability for N vicious walkers moving on an infinite line. For large times, $Q(\mathbf{x}, t)$ decays as a power:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{4}} \ . \tag{2.1}$$

Interesting results also arise when further conditions are imposed on the movement of the vicious walkers by the use of absorbing or reflecting walls, where all walkers are initially located on the same side of the boundary (the case where there are walkers on both sides decouples into two independent problems). While Fisher [2] found the survival probability of vicious walkers with an absorbing boundary at the origin only for $N = 2$, Krattenthaler et al. [20] were able to determine the exact asymptotic form for N vicious walkers starting from equi-spaced lattice points:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N^2}{2}}. \tag{2.2}$$

By evaluating the scaling limit Katori and Tanemura [21] showed that this asymptotic behaviour holds for arbitrary initial positions on a continuous line. These results were found even earlier by Forrester [22] by generalising Fisher's approach to the case of N random walkers.

In this chapter we consider the interesting problem of N vicious walkers moving in an attractive one-body potential $v(x)$, i.e. the full potential function has the separable form $V(\mathbf{x}) = \sum_{i=1}^{N} v(x_i)$. Treating both time and space as continuous, we investigate the survival probability of N vicious walkers with equal diffusion constants D. The equation of motion for walker i is taken to be

$$\dot{x}_i = -\frac{\partial V}{\partial x_i} + \eta_i(t), \tag{2.3}$$

where the Langevin noise $\eta_i(t)$ is a Gaussian white noise with mean zero and correlator

$$\langle \eta_i(t) \eta_j(t') \rangle = 2D\delta_{ij}\delta(t - t'). \tag{2.4}$$

For a square-well potential of width L we consider three different combinations of absorbing and reflecting walls and find an exponential decay for the survival probability of the general form $Q(\mathbf{x}, t) \sim e^{-\theta_N t}$. For two reflecting walls the exponent θ_N is determined to be

$$\theta_N^{RR} = D\frac{\pi^2}{L^2}\frac{N(N-1)(2N-1)}{6}. \tag{2.5}$$

In the case of one reflecting and one absorbing wall we obtain

$$\theta_N^{RA} = D\frac{\pi^2}{L^2}\frac{N(2N+1)(2N-1)}{12}, \tag{2.6}$$

while for two absorbing walls the exponent of the asymptotic decay is:

$$\theta_N^{AA} = D\frac{\pi^2}{L^2}\frac{N(N+1)(2N+1)}{6}. \tag{2.7}$$

An interesting potential, which turns out to be a powerful tool, is the problem of N vicious walkers in the harmonic potential $V(\mathbf{x}) = \frac{a}{2}\mathbf{x}^2$. The asymptotic behaviour for large times is determined to be an exponential decay independent of the diffusion constant:

$$\theta_N = \frac{N(N-1)}{2}a. \tag{2.8}$$

This result also provides a mechanism to determine the survival probability of N vicious walkers on an infinite line in a simple way. By mapping the zero-potential problem to the harmonic potential problem, we derive Fisher's result [2] and also the result by Forrester and Krattenthaler et al [22, 20] with an absorbing wall at the origin. Furthermore, we are able to obtain, to our knowledge, a new result for the survival probability of N vicious walkers on a semi-infinite line with a reflecting boundary at the origin [1, 23], which decays as:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{2}}. \tag{2.9}$$

This chapter is organised as follows. In the first section the method for a general one-body potential $v(x)$ is presented, while in the second section explicit results for square-well and harmonic potentials are given. Afterwards the case of zero potential is revisited, obtaining the known results, and a new result for a system with a reflecting boundary, through a transformation to the harmonic problem. At the end a short conclusion is given.

2.2 The method

The dynamics of a random walker, with position coordinate x_i, moving in a potential $V(\mathbf{x})$ is described, in continuous space and time, by the Langevin equation (2.3) with noise correlator (2.4).

The probability $Q(\mathbf{x}, t)$ that all N vicious walkers, $i = 1, \ldots, N$, have survived up to time t, given that they *started* at $\{x_i\}$, satisfies the corresponding backward Fokker-Planck equation:

$$\frac{\partial Q(\mathbf{x}, t)}{\partial t} = D \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} Q(\mathbf{x}, t) - \sum_{i=1}^{N} \frac{\partial V(\mathbf{x})}{\partial x_i} \frac{\partial Q(\mathbf{x}, t)}{\partial x_i}. \tag{2.10}$$

For convenience we start by defining the survival probability $q(x_i, t)$ of just one random walker moving in a potential restricted by the imposed boundary conditions. This survival probability $q(x_i, t)$ satisfies the backward Fokker-Planck equation

$$\frac{\partial q(x_i, t)}{\partial t} = D \frac{\partial^2}{\partial x_i^2} q(x_i, t) - \frac{dv(x_i)}{dx_i} \frac{\partial q(x_i, t)}{\partial x_i}, \tag{2.11}$$

where we have used the relation $V(\mathbf{x}) = \sum_i v(x_i)$ for a one-body potential. For any such potential, the backward Fokker-Planck equation (2.11) is separable in time and space. Let us call these separable solutions, i.e. the solutions of equation (2.11) satisfying the relevant boundary conditions, single-walker basis functions. They have the form $q_j(x_i, t) = u_j(x_i) \exp(-\lambda_j t)$, where λ_j is the decay rate associated with basis function j, and these rates are ordered such that $\lambda_1 < \lambda_2 < \lambda_3 \ldots$.

For N non-interacting walkers moving in the same potential, the N-walker basis functions for the survival probability take the form of products of N single-walker functions, each with a different space variable x_i. Since, however, we are investigating vicious walkers the mutual annihilation property must be respected. Since two walkers die when arriving at the same x-coordinate, the boundary condition $Q(x_1, \ldots, x_n, t) = 0$ when $x_i = x_j$ for any $i \neq j$ must be respected. This property is ensured by constructing $Q(\mathbf{x}, t)$ using antisymmetric combinations of products of N single-walker functions, analogous to the antisymmetric construction of the wavefunction of fermions [2]. The N-walkers basis functions of the vicious walkers problem with N walkers have, therefore, the form

$$Q^{i_1, \ldots, i_N}(\mathbf{x}, t) = \det A^{i_1, \ldots, i_N}, \tag{2.12}$$

where the elements of the $N \times N$ matrix A are given by

$$A_{nm}^{i_1, \ldots, i_N} = q_{i_n}(x_m, t). \tag{2.13}$$

The full solution $Q(\mathbf{x},t)$ is a linear superposition of these basis functions with coefficients determined by the initial condition.

To solve the problem of N vicious walkers in an arbitrary potential, therefore, we need to find the single-walker basis functions $q_j(x_i,t)$ appropriate to the imposed boundary conditions, introduced in chapter 1. For an absorbing boundary at $x=a$ the functions $q_j(x_i,t)$ must satisfy

$$q_j(x_i=a,t)=0. \tag{2.14}$$

For a reflecting boundary at $x=b$ the boundary condition for the backward Fokker-Planck equation is in the one-dimensional case:

$$\left.\frac{dq_j}{dx_i}\right|_{x_i=b}=0. \tag{2.15}$$

Clearly these boundary conditions are also satisfied by the functions $Q^{i_1,\ldots,i_N}(\mathbf{x},t)$, since the latter is just an antisymmetrised product of single-walker basis functions.

Consider now the late-time limit, $t\to\infty$. Each antisymmetrised product in the expression for $Q(\mathbf{x},t)$ contains N different relaxation factors $\exp(-\lambda_j t)$. The slowest-decaying term in the sum, therefore, is the term in which the relaxation rates are $\lambda_1,\lambda_2,\ldots,\lambda_N$. It follows that, asymptotically,

$$Q(\mathbf{x},t)\propto \det B^{1,2,\ldots,N}\exp(-\theta_N t), \tag{2.16}$$

where $B^{1,2,\ldots,N}$ is just the $N\times N$ matrix with elements $B_{nm}=u_n(x_m)$ $(n,m=1,\ldots,N)$, i.e. it is constructed using the N slowest-decaying single-walker basis functions, and the total decay rate is

$$\theta_N=\sum_{j=1}^{N}\lambda_j\ . \tag{2.17}$$

The following sections provide some applications of this general result.

2.3 Results for vicious walkers in a potential

In this section we discuss two examples of N vicious walkers in a potential and determine the decay of the survival probability $Q(\mathbf{x},t)$.

2.3.1 The square-well potential

Consider a square-well potential which has two walls of infinite potential, one at the origin and the other at $x = L$, and vanishes between the walls. A vicious walker restricted to move between the walls satisfies the backward Fokker-Planck equation:

$$\frac{\partial q(x_i,t)}{\partial t} = D\frac{\partial^2 q(x_i,t)}{\partial x_i^2}. \tag{2.18}$$

This equation can be solved in general by separation of variables, which amounts in this case to writing the solution as a spatial Fourier series. Different solutions result from the various sets of boundary conditions imposed by the property of the walls.

Two reflecting walls

For two reflecting walls the spatial derivative of $q(x_i,t)$ must be zero at $x = 0$ and $x = L$. In this case, therefore $q(x_i,t)$ is given by Fourier cosine series with basis functions

$$q_n(x_i,t) = \exp\left(-\frac{n^2\pi^2 Dt}{L^2}\right)\cos\left(\frac{\pi}{L}nx_i\right), \quad n = 0, 1, \ldots \tag{2.19}$$

The survival probability is constructed as a superposition of antisymmetrised products of these basis functions:

$$\begin{aligned} Q(\mathbf{x},t) &= \sum_{i_1}\cdots\sum_{i_N} C^{i_1,\ldots,i_N}\det A^{i_1,\ldots,i_N} \\ &= \sum_{i_1}\cdots\sum_{i_N} C^{i_1,\ldots,i_N}\exp\left(-\frac{\pi^2 Dt}{L^2}\sum_{n=1}^{N} i_n^2\right) \\ &\quad\times\det B^{i_1,\ldots,i_N}, \end{aligned} \tag{2.20}$$

where

$$B_{nm}^{i_1,\ldots,i_N} = \cos\left(\frac{\pi}{L}i_n x_m\right). \tag{2.21}$$

To evaluate the long-time behaviour we keep only the N longest-lived modes, given by the N smallest values, $i = 0, 1, \ldots, N-1$ of i_n. Using $\sum_{i=0}^{N-1} i^2 = N(N-1)(2N-1)/6$ we obtain, for the asymptotic time-dependence,

$$Q(\mathbf{x},t) \sim \exp\left(-\frac{\pi^2 Dt}{L^2}\,\frac{N(N-1)(2N-1)}{6}\right). \tag{2.22}$$

One reflecting and one absorbing wall

For an absorbing wall at the origin and a reflecting wall at $x = L$ the boundary conditions are satisfied by a Fourier sine series with basis functions:

$$\begin{aligned} q_n(x_i, t) &= \exp\left(-\frac{(2n+1)^2\pi^2 Dt}{4L^2}\right) \\ &\times \sin\left(\frac{\pi}{2L}(2n+1)x_i\right), \quad n = 0, 1, \ldots \end{aligned} \tag{2.23}$$

Analogous to the preceding case the survival probability for all N vicious walkers is constructed and the asymptotic survival probability for large time is evaluated using $\sum_{i=0}^{N-1}(2i+1)^2 = N(2N+1)(2N-1)/3$ to give the asymptotic decay

$$Q(\mathbf{x}, t) \sim \exp\left(-\frac{\pi^2 Dt}{L^2}\frac{N(2N+1)(2N-1)}{12}\right). \tag{2.24}$$

Two absorbing walls

In the case of two absorbing walls the basis functions have to vanish at both $x = 0$ and $x = L$. A Fourier sine series is therefore appropriate, with basis functions

$$q_n(x_i, t) = \exp\left(-\frac{n^2\pi^2 Dt}{L^2}\right)\sin\left(\frac{\pi}{L}nx_i\right), \quad n = 1, 2, \ldots \tag{2.25}$$

This is very similar to the result for two reflecting boundaries, except that the spatial functions are sines so the sum begins with $n = 1$. The large-time behaviour of $Q(\mathbf{x}, t)$ is given by

$$Q(\mathbf{x}, t) \sim \exp\left(-\frac{\pi^2 Dt}{L^2}\frac{N(N+1)(2N+1)}{6}\right). \tag{2.26}$$

Before proceeding to the harmonic potential, we note that the inequalities $2N(N-1)(2N-1) < N(2N+1)(2N-1) < 2N(N+1)(2N+1)$, for all $N \geq 1$, imply that for a well of given size the decay is fastest with two absorbing boundaries and slowest with two reflecting boundaries, as is intuitively clear.

2.3.2 The harmonic potential

A harmonic potential $V(\mathbf{x}) = \frac{a}{2}\mathbf{x}^2$ is considered for which the backward Fokker-Planck equation for the single-walker basis function reads

$$\frac{\partial q(x_i, t)}{\partial t} = D\frac{\partial^2}{\partial x_i^2}q(x_i, t) - a\, x_i\frac{\partial q(x_i, t)}{\partial x_i}. \tag{2.27}$$

This equation can be transformed into an imaginary-time Schrödinger equation by the substitution $q(x_i,t) = \exp(ax_i^2/4D)\psi(x_i,t)$ to give

$$\frac{\partial\psi(x_i,t)}{\partial t} = D\frac{\partial^2}{\partial x_i^2}\psi(x_i,t) + \left(\frac{a}{2} - \frac{a^2x_i^2}{4D}\right)\psi(x_i,t). \tag{2.28}$$

This equation has solutions of the form $\psi(x_i,t) = e^{-\lambda t}u(x_i)$, where $u(x_i)$ satisfies the ordinary differential equation

$$\left(D\frac{d^2}{dx_i^2} + \left(\frac{a}{2} - \frac{a^2x_i^2}{4D}\right)\right)u(x_i) = -\lambda u(x_i). \tag{2.29}$$

This equation is equivalent to the time-independent Schrödinger equation for the harmonic oscillator. The eigenvalues and eigenfunctions of this eigenvalue problem are well known: see for example a similar problem in reference [24]. The eigenfunctions have the form

$$u_n(x_i) = H_n\left(x_i\sqrt{\frac{a}{2D}}\right)\exp(-\frac{a}{4D}x_i^2), \tag{2.30}$$

where the functions $H_n(x)$ are the Hermite polynomials defined by

$$H_n(y) = (-1)^n e^{y^2}\frac{d^n}{dy^n}e^{-y^2}. \tag{2.31}$$

The corresponding eigenvalues are $\lambda_n = na$, where $n = 0, 1, 2, \ldots$. The original basis functions $q(x_i,t)$ are given by $q_n(x_i,t) = H_n\left(x_i\sqrt{\frac{a}{2D}}\right)\exp(-\lambda_n t)$.

Applying the antisymmetrisation process to determine the survival probability of N vicious walkers in a harmonic potential we obtain the asymptotic time dependence:

$$Q(\mathbf{x},t) \sim \exp\left(-at\sum_{i=0}^{N-1} i\right) \tag{2.32}$$

giving

$$Q(\mathbf{x},t) \sim \exp\left(-at\,\frac{N(N-1)}{2}\right). \tag{2.33}$$

This result can be checked in the case $N = 2$ with the full solution for the survival probability of two vicious walkers in a harmonic potential derived in section 1.4.2. Using the method of antisymmetrised basis functions the asymptote of the survival probability $Q(x_1,x_2,t)$ of two vicious walkers with initial positions x_1 and x_2, respectively, where $x_1 \leq x_2$, is given by the antisymmetrised product of the two slowest decaying basis functions: $Q(x_1, x_2, t) = q_0(x_1, t)q_1(x_2, t) - q_1(x_1, t)q_0(x_2, t)$. Substituting the solution of the basis functions for the harmonic potential yields:

$$Q(x_1, x_2, t) \sim e^{-at} \left[H_0 \left(x_1 \sqrt{\frac{a}{2D}} \right) H_1 \left(x_2 \sqrt{\frac{a}{2D}} \right) - H_1 \left(x_1 \sqrt{\frac{a}{2D}} \right) H_0 \left(x_2 \sqrt{\frac{a}{2D}} \right) \right], \tag{2.34}$$

where the first two Hermite polynomials are defined as $H_0(x) = 1$ and $H_1(x) = 2x$. Introducing the relative coordinate $x_{12} = x_2 - x_1$ gives the survival probability for large times:

$$Q(x_{12}, t) \sim x_{12} e^{-at}. \tag{2.35}$$

This solution is equivalent to the former result in section 1.4.2.

The general solutions for N vicious walkers in a harmonic potential can readily be extended to the case where there is a reflecting or absorbing boundary at $x = 0$ and all the walkers start on the same side of the boundary (if there are walkers on both sides, the problem decouples into two independent problems). For a reflecting boundary, the boundary condition $u'(0) = 0$ selects only the even-numbered Hermite polynomials, $n = 0, 2, 4, \ldots$, and

$$\begin{aligned} Q(\mathbf{x}, t) &\sim \exp\left(-at \sum_{i=0}^{N-1} 2i \right) \\ &= \exp[-at\, N(N-1)] \text{ (reflecting wall).} \end{aligned} \tag{2.36}$$

For an absorbing boundary, the boundary condition $u(0) = 0$ selects the odd-numbered Hermite polynomials to give

$$\begin{aligned} Q(\mathbf{x}, t) &\sim \exp\left(-at \sum_{i=1}^{N} (2i-1) \right) \\ &= \exp[-at\, N^2] \text{ (absorbing wall).} \end{aligned} \tag{2.37}$$

In the following section we show how these results can be used to compute the survival probability of N vicious walkers in *zero* potential, with and without an absorbing or reflecting wall, by mapping the problem back to the oscillator problem.

2.4 Vicious walkers on a line

Here the case of N vicious walkers restricted by no potential is investigated. This problem can be solved in a quite simple way by mapping it to the problem of N vicious walkers in a harmonic potential and using the previous results. Again, we consider the Langevin equation (2.3), but with $V(\mathbf{x}) = 0$, and let all N vicious walkers start to move at time $t = t_0$. We introduce the following mapping from $\mathbf{x}, t$ to the new coordinates $\mathbf{X}, T$ by [25, 26]:

$$\mathbf{X} = \frac{\mathbf{x}}{\sqrt{2Dt}}, \qquad t = t_0\, e^T. \tag{2.38}$$

Then the Langevin equation (2.3) transforms to

$$\frac{dX_i(T)}{dT} = -\frac{1}{2} X_i(T) + \xi_i(T), \tag{2.39}$$

where $\xi_i(T) = \sqrt{t_0/2D}\, e^{T/2}\, \eta_i(t_0\, e^T)$ is a Gaussian white noise with mean zero and correlator

$$\langle \xi_i(T) \xi_j(T') \rangle = \delta_{ij} \delta(T - T'). \tag{2.40}$$

The corresponding backward Fokker-Planck equation in the new coordinates is

$$\frac{\partial Q(\mathbf{X}, T)}{\partial T} = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial X_i^2} Q(\mathbf{X}, T) - \frac{1}{2} \sum_{i=1}^{N} X_i \frac{\partial Q(\mathbf{X}, T)}{\partial X_i} \tag{2.41}$$

where the space coordinates are now the starting points of the vicious walkers, given by:

$$X_i(T = 0) = \frac{x_i(t_0)}{\sqrt{2Dt_0}}\ . \tag{2.42}$$

In the new coordinates this problem looks identical to the harmonic potential problem with $a = 1/2$ and $D = 1/2$. Hence the asymptotic (in time) solution for the survival probability of N vicious walkers is, according to our previous results,

$$Q(\mathbf{X}, T) \sim \exp\left(-\frac{T}{2} \sum_{i=0}^{N-1} i \right) \det B^H, \tag{2.43}$$

where $(B^H)_{nm} = H_{n-1}(X_m/\sqrt{2})$ and $n, m = 1, \ldots, N$. Mapping back to the original coordinates $(\mathbf{x}, t)$ leads to the asymptotic survival probability

$$Q(\mathbf{x}, t) \sim \left(\frac{t}{t_0} \right)^{-\frac{1}{2} \sum_{i=0}^{N-1} i} \det B^L, \tag{2.44}$$

where $(B^L)_{nm} = H_{n-1}(x_m/2\sqrt{Dt_0})$, with $n, m = 1, \ldots, N$. The determinant $\det B^L$ is proportional to the Vandermonde determinant [2]: $\det B^L = (Dt_0)^{-N(N-1)/4} \prod_{i<j} |x_i - x_j|$, and all t_0-dependence drops out, as it must, to give the long-time behaviour

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{4}}. \tag{2.45}$$

which is just the result Fisher obtained [2]. But our approach to the problem also gives a simple way to obtain expressions for the survival probability for N vicious walkers with an absorbing or reflecting wall at the origin (and all walkers starting on one side of the wall).

The essential arguments have been given in the preceding subsection. For an absorbing (reflecting) boundary, only the odd (even) basis functions contribute. Note first that the Fisher result (2.45) follows immediately from (2.33) on setting $a = 1/2$ and $T = \ln(t/t_0)$. The detailed discussion above was given mainly to show how the arbitrary scale t_0 drops out. To obtain the asymptotic results for a reflecting or absorbing wall at the origin, we can simple make the same replacements in equations (2.36) and (2.37) respectively. For the absorbing boundary, we recover the result of Krattenthaler et al. [20]:

$$Q(\mathbf{x}, t) \sim t^{-\frac{N^2}{2}} \quad \text{(absorbing wall)}, \tag{2.46}$$

while for a reflecting wall we obtain

$$Q(\mathbf{x}, t) \sim t^{-\frac{N(N-1)}{2}} \quad \text{(reflecting wall)}. \tag{2.47}$$

The latter is, to our knowledge, a new result.

Shortly after the paper including this new result was submitted for publication we learnt that it had been obtained independently, using a different method, by M Katori and H Tanemura [23].

As a final comment we note that the case where the absorbing or reflecting wall moves, with a displacement $x_w = ct^{1/2}$, is also amenable in principle to exact analysis. The change of variable (2.38) maps the problem to one where the N walkers move in a harmonic oscillator potential, and the absorbing or reflecting wall is at a *fixed position* in the new coordinates. This problem has been analysed for a single walker [27, 28, 29], and the survival probability decays as $t^{-\theta}$, where the exponent θ is found to vary continuously with the amplitude, c, of the wall displacement. The same qualitative features will be present for N vicious walkers. For a reflecting (R) or absorbing (A) boundary, one will obtain a decay exponent $\theta_{R,A} = N(N-1)/4 + f_{R,A}(c, N)$, where $f_{R,A}(-\infty, N) = 0$, corresponding to a rapidly receding wall, which will be equivalent to no wall at all, and $f_R(0, N) = N(N-1)/4$, $f_A(0, N) = N(N+1)/4$ corresponding to a static wall at the origin.

2.5 Summary

In this chapter we have derived the exact asymptotes for the survival probability of vicious walkers moving in a square well potential and a harmonic potential with various combinations of absorbing and reflecting walls. The results for a harmonic potential have been used to find the properties of free vicious walkers (zero potential) through a change of variables, and a new result obtained for the case of a single reflecting boundary. Comparing all results for each potential one recognises that the survival probability decays faster when the number of walls is increased, with absorbing walls causing a faster decrease than reflecting walls, in accordance with intuitive expectations.

Chapter 3

Vicious walkers in an inverted harmonic potential

In this chapter we derive the exact form of the survival probability of three vicious walkers in an inverted harmonic potential in the limit of infinite time. To obtain this result we map the problem of three one-dimensional vicious walkers to a single random walker in two-dimensional wedge, where the single random walker is eliminated on hitting the boundaries of the wedge. To illustrate the mapping diffusion processes in a two-dimensional wedge are introduced, reviewing known results for the survival probability of a random walker in a wedge and three one-dimensional vicious walkers with different diffusion constants.

3.1 Introduction

The geometry of a two-dimensional wedge with absorbing boundaries has proved to be of interest in itself and as a mean to study one-dimensional interacting random walks.

The survival probability of a random walker in a wedge decays as a power law where the exponent only depends on the opening angle Θ of the wedge (Figure 3.1):

$$Q(x, y, t) \sim t^{-\frac{\pi}{2\Theta}} \tag{3.1}$$

This is due to the absence of other characteristic length or time scales in the wedge with boundaries infinite in radius [14].

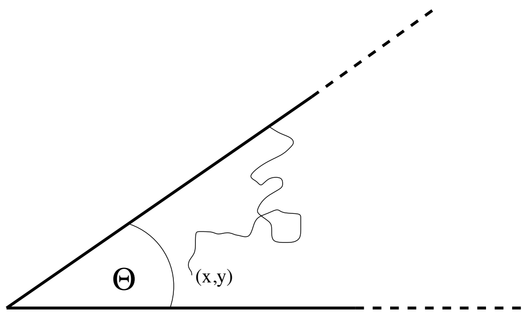

Figure 3.1: A random walker diffusing in a wedge with absorbing boundaries and opening angle Θ.

To solve complicated diffusion processes of interacting walkers like vicious walkers in one-dimension the stochastic process in one-dimension is mapped onto the two-dimensional wedge, where the wedge boundaries correspond to the interaction interfaces [30, 31]. By this means Fisher and Gelfand [31] calculated the survival probability of three vicious walkers on a line with different diffusion constants. The probability that all three vicious walkers with initial positions x_1, x_2, x_3 and diffusion constants D_1, D_2, D_3 have survived up to time t decays for large times t as $Q(x_1, x_2, x_3, t) \sim t^{-\theta}$, where

$$\theta = \frac{\pi}{2\cos^{-1}\left[\frac{D_2}{\sqrt{(D_1+D_2)(D_2+D_3)}}\right]}. \tag{3.2}$$

The problem of N vicious walkers with different diffusion constants remains intractable for $N \geq 4$. Krapivsky and Redner extended Fisher's idea to the lion and lamb problem, in which lions eliminate the lamb on contact but are indifferent among each other [32, 33].

In our new problem we calculate the infinite time survival probability of three one-dimensional vicious walkers with identical diffusion constants moving in an inverted harmonic potential by mapping the problem to a diffusion process in a wedge. In terms of the relative coordinates of the three walkers $y_1 = x_2 - x_1$ and $y_2 = x_3 - x_2$ the infinite time survival probability can be expressed as an infinite sum over confluent hypergeometric functions of the first kind. Graphs are presented giving the infinite time survival probability for all relative initial positions of the three vicious walkers.

The outline of this chapter is as follows. First the survival probability of a random walker in a two-dimensional wedge will be investigated by use of scaling concepts. Afterwards the one-dimensional random processes are considered. In section 3.3.1 a review of three vicious walkers with different diffusion constants is given. In section 3.3.2 explicit results for the infinite time survival probability of vicious walkers in an inverted potential are presented.

3.2 Random walker in a wedge

In this section we calculate the asymptotic decay of the survival probability of a diffusion process in a wedge with absorbing boundaries. Our derivation is based on a very elegant method introduced by Burkhardt [34] which uses scaling properties to derive the behaviour of stochastic processes for large times.

To model a diffusion process we consider a two-dimensional random walker in a wedge of opening angle Θ that evolves according to the Langevin equation:

$$\dot{x}_i = \eta_i(t), \quad i = 1, 2, \tag{3.3}$$

where $\eta_i(t)$, $i = 1, 2$, is a Gaussian white noise with zero mean and correlator $\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$. To calculate the survival probability $Q(x_1, x_2)$ the corresponding backward Fokker-Planck equation is examined. In accordance to the symmetry of the problem, polar coordinates (r, φ) are chosen. In those variables the boundary conditions reduce to $Q(r, \varphi = 0) = 0$ and $Q(r, \varphi = \Theta) = 0$. Since the random walker moves in zero potential the backward Fokker-Planck equation is just the diffusion equation:

$$\frac{\partial Q(r,\varphi,t)}{\partial t} = D\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) Q(r,\varphi,t). \tag{3.4}$$

All terms on the right hand side of this equation have dimensions $\frac{D}{r^2}$ and the term on the left hand side has dimension $\frac{1}{t}$, hence we obtain $r^2 \sim Dt$. If we choose $z = r^2/Dt$ as our dimensionless scaling variable the number of independent variables of the survival probability is reduced to two:

$$Q(r,\varphi,t) = \mathcal{Q}\left(\frac{r^2}{Dt}, \varphi\right). \tag{3.5}$$

Since we are expecting a power law decay for large times of the form $t^{-\theta}$, the survival probability must have an asymptotic behaviour as:

$$Q(r, \varphi, t) \sim \left(\frac{r^2}{Dt}\right)^{\theta} F(\varphi), \tag{3.6}$$

for large times t.

Inserting (3.6) in the backward Fokker-Planck equation (3.4) gives:

$$-\frac{\theta}{Dt} F(\varphi) = \frac{2\theta(2\theta - 1)}{r^2} F(\varphi) + \frac{2\theta}{r^2} F(\varphi) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} F(\varphi). \tag{3.7}$$

Obviously the term on the left hand side is of order $1/Dt$ and is, therefore, subdominant in the large time behaviour. Omitting this term the differential equation for the function $F(\varphi)$ becomes:

$$4\theta^2 F(\varphi) + \frac{\partial^2}{\partial \varphi^2} F(\varphi) = 0. \tag{3.8}$$

To find a solution satisfying the given boundary conditions we select the sine modes:

$$F(\varphi) = \sum_{n=1}^{\infty} A_n \sin(2\theta_n \varphi), \tag{3.9}$$

where $\theta_n = \frac{n\pi}{2\Theta}$.

For the large time behaviour we are only interested in the first, slowest decaying mode, $n = 1$. Hence the survival probability for a random walker in an absorbing wedge of opening angle Θ decays as

$$Q(r, \varphi, t) \sim t^{-\frac{\pi}{2\Theta}}. \tag{3.10}$$

Note that the survival probability for $\Theta = 2\pi$, an absorbing half-line, decays as $t^{-1/4}$.

This problem has also been investigated in three dimensions, i.e. a random walker in an absorbing cone [14]. In the limit of small opening angles the survival probability also decays as a power law, with a power proportional to the inverse of the opening angle. However, in the limit of large opening angle this is not true anymore and in the case of opening angle equal to 2π the survival probability is even finite. This example just shows a transition between recurrence and transience of a random walk. In two dimensions a random walker will always a hit an absorbing half-line, whereas in three dimensions there is a finite probability that it does not.

3.3 One-dimensional problems solved in the wedge

Problems of three one-dimensional interacting random walkers are accessible by mapping the process to a single random walker in an appropriately chosen two-dimensional wedge. For this purpose the individual coordinates of the three walkers x_1, x_2 and x_3 are regarded as the coordinates of a single random walker in three dimensions, which are projected down to a two-dimensional wedge in the space of relative coordinates [14]. The wedge boundaries correspond to the interacting interfaces.

At first the well-studied problem of three vicious walkers on a line [30, 31, 32, 33] will be investigated, followed by the new problem of three vicious walkers in an inverted harmonic potential.

3.3.1 Vicious walkers with different diffusion constants

We consider three vicious walker moving on a line with initial positions x_1, x_2, x_3 and different diffusion constants D_1, D_2 and D_3. To obtain the asymptotic behaviour of the survival probability for large times t, the process is mapped onto the diffusion of a single random walker in a two-dimensional wedge, with an opening angle Θ, that depends on the ratio of the diffusion constants. Since the survival probability of a single random walker is known to decay as $Q(x, y, t) \sim t^{-\frac{\pi}{2\Theta}}$, see equation (3.10), calculating the opening angle is sufficient to solve the problem.

All three vicious walkers are still alive if they retain their ordering $x_1 < x_2 < x_3$. By choosing isotropic coordinates defined by $y_i = x_i/\sqrt{D_i}$ this constraint can also be written as $y_1\sqrt{D_1} < y_2\sqrt{D_2} < y_3\sqrt{D_3}$. Interpreting the isotropic coordinates y_i, $i = 1, 2, 3$, as the three-dimensional coordinates of a single random walker, the single walker stays alive if it does not hit one of the planes given by:

$$y_1\sqrt{D_1} = y_2\sqrt{D_2}, \quad y_2\sqrt{D_2} = y_3\sqrt{D_3}. \tag{3.11}$$

These two plane intersect, including a wedge in which the single random walker is diffusing, see figure 3.2. To obtain the opening angle of this wedge the unit normals

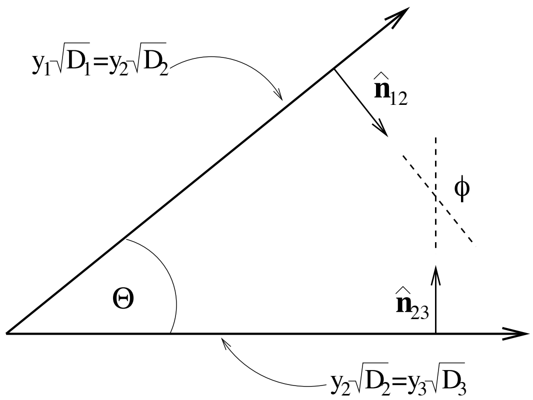

Figure 3.2: The intersection of the two absorbing planes creates a wedge of opening angle Θ in which the single random walker diffuses.

$\hat{\mathbf{n}}_{12}$ and $\hat{\mathbf{n}}_{23}$ to the planes $y_1\sqrt{D_1} = y_2\sqrt{D_2}$ and $y_2\sqrt{D_2} = y_3\sqrt{D_3}$ are computed [14, 31]:

$$\hat{\mathbf{n}}_{12} = (-\sqrt{D_1}, \sqrt{D_2}, 0)/\sqrt{D_1 + D_2} \tag{3.12}$$

$$\hat{\mathbf{n}}_{23} = (0, -\sqrt{D_2}, \sqrt{D_3})/\sqrt{D_2 + D_3} \tag{3.13}$$

The angle ϕ included by the unit normals corresponds to an opening angle $\Theta = \pi - \phi$. Hence evaluating the scalar product of the unit normals gives the opening angle:

$$\Theta = \cos^{-1}\left[\frac{D_2}{\sqrt{(D_1 + D_2)(D_2 + D_3)}}\right], \tag{3.14}$$

and so the survival probability decays as $Q(x_1, x_2, x_3, t) \sim t^{-\theta}$ with θ defined by:

$$\theta = \frac{\pi}{2\cos^{-1}\left[\frac{D_2}{\sqrt{(D_1+D_2)(D_2+D_3)}}\right]}. \tag{3.15}$$

A full solution of the survival probability is given by Fisher and Gelfand [31].

The result for the exponent θ in equation (3.15) can easily be checked with special cases of results in chapter 2. For three vicious walkers with equal diffusion constants the above exponent reduces to $\theta = 3/2$, which corresponds to the solution of equation (2.45) in section 2.4, $\theta = N(N-1)/4$ and $N = 3$. Also setting $x_1 = 0$ and $D_1 = 0$ with $D_2 = D_3$ gives a familiar result from section 2.4. This is the case of $N = 2$ vicious walkers on one side of an absorbing wall at the origin, as solved for N walkers in equation (2.46): $\theta = N^2/2 = 2$. In the case of $x_2 = 0$ and $D_2 = 0$ the problem decouples into two independent vicious walkers with an absorbing wall at the origin, yielding a power law decay with exponent: $\theta = 2(N^2/2) = 1$ for $N = 1$.

3.3.2 Vicious walkers in an inverted potential

After reviewing the concept and ideas of the two-dimensional wedge, we now consider the new problem of three one-dimensional vicious walkers, with initial coordinates $x_1 < x_2 < x_3$, in an inverted harmonic one-body potential, $v(x_i) = -\frac{a}{2}x_i^2$, with a positive. By mapping this process onto a single random walker in a wedge with absorbing boundaries, we calculate the infinite time survival probability $Q(x_1, x_2, x_3)$. In an inverted harmonic potential the vicious walkers are subjected to a drift, which separates them, therefore, there is a non-zero survival probability for large times.

The equation of motion of each random walker is taken to be:

$$\dot{x}_i = ax_i + \eta_i(t), \quad i = 1, 2, 3, \tag{3.16}$$

where $\eta_i(t)$ is a Gaussian white noise with zero mean and correlator $\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$. Mapping this process onto a single two-dimensional random walker in a wedge, we introduce relative coordinates $y_1 = x_2 - x_1$, and $y_2 = x_3 - x_2$. This single random walker now obeys the following equation of motion:

$$\dot{y}_j = ay_j + \xi_j, \quad j = 1, 2, \tag{3.17}$$

where ξ_j is the 'relative' Gaussian white noise defined by $\xi_1 = \eta_2 - \eta_1$ and $\xi_2 = \eta_3 - \eta_2$. The mean is zero as beforehand but the correlator now becomes

$$\langle \xi_i(t)\xi_j(t')\rangle = \begin{cases} 4D\delta(t-t') & \text{for } i = j \\ -2D\delta(t-t') & \text{for } i \neq j \end{cases}$$

To determine the infinite time survival probability of the single two-dimensional random walker we consider the time-independent backward Fokker-Planck equation in the initial coordinates y_1, y_2:

$$a\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)Q(y_1, y_2) + 2D\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\right)Q(y_1, y_2) = 0. \tag{3.18}$$

The variables are chosen to be dimensionless by rescaling $\tilde{y}_i = y_i\sqrt{a/2D}$, $i = 1, 2$.

Since vicious walkers annihilate on meeting, the survival probability of the single random walker must vanish at $\tilde{y}_1 = 0$ and $\tilde{y}_2 = 0$ corresponding to the absorbing boundaries of a wedge with opening angle $\Theta = \pi/2$, in which the single random walker is diffusing, see figure 3.3. If the vicious walkers are infinitely far apart, the survival probability should be unity, hence $Q(\tilde{y}_1 = \infty, \tilde{y}_2) = Q(\tilde{y}_1, \tilde{y}_2 = \infty) = 1$.

In order to reduce equation (3.18) to a canonical form, a change of variables is required. Introducing the new variables u and v according to

$$\tilde{y}_1 = \frac{u + \sqrt{3}v}{2} \quad \tilde{y}_2 = \frac{u - \sqrt{3}v}{2}, \tag{3.19}$$

transforms equation (3.18) to:

$$\left[u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right] Q(u, v) = 0. \tag{3.20}$$

The absorbing boundaries in the new variables u and v are at $v = \pm\sqrt{3}u$. Therefore,

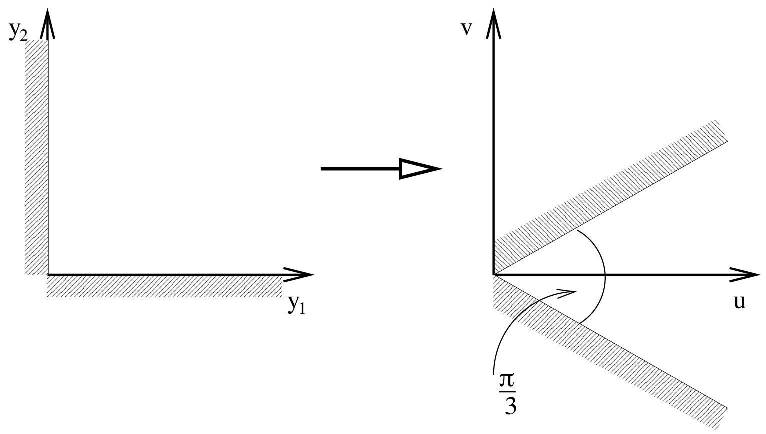

Figure 3.3: The transformation to a canonical differential equation maps the right angled wedge in (y_1, y_2) coordinates to an axisymmetric wedge of opening angle $\Theta = \pi/3$.

in the new variables the wedge is axis-symmetric about the u-axis and has an opening angle of $\Theta = \pi/3$, see figure 3.3. Because of the symmetry of the wedge, polar coordinates (r, φ) are appropriate. Hence the time-independent backward Fokker-Planck equation becomes:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \left(\frac{1}{r} + r \right) \frac{\partial}{\partial r} \right] Q(r, \varphi) = 0. \tag{3.21}$$

The boundary conditions reduce to $Q(r, \varphi = \pi/6) = Q(r, \varphi = -\pi/6) = 0$ and $Q(r = 0, \varphi) = 0$ at the absorbing boundaries of the wedge and $Q(r = \infty, \varphi) = 1$ corresponding to the survival of all vicious walkers if they are initially at infinite distance from each other.

The partial differential equation (3.21) can be solved by separation of variables $Q(r,\varphi) = \sum_{n=1}^{\infty} A_n R_n(r)\Phi_n(\varphi)$, where the angular part $\Phi_n(\varphi)$ is a cosine mode satisfying the angular boundary conditions:

$$\Phi_n(\varphi) = \cos(3(2n-1)\varphi), \tag{3.22}$$

and the A_n are constant coefficients to be determined by the boundary conditions.

Substituting the result for $\Phi_n(\varphi)$ in (3.21) yields the following ordinary differential equation for $R_n(r)$.

$$r^2 R_n''(r) + \left(r + r^3\right) R_n'(r) - 9(2n-1)^2 R_n(r) = 0 \tag{3.23}$$

By setting $r^2 = \zeta$ and $R_n(r) = \zeta^{3n-\frac{3}{2}}\rho_n(\zeta)$ this differential equation is transformed into, see 2.215 in reference [35]:

$$\zeta\rho_n''(\zeta) + \left(\frac{1}{2}\zeta + 6n - 2\right)\rho_n'(\zeta) + \frac{6n-3}{4}\rho_n(\zeta) = 0. \tag{3.24}$$

This ordinary differential equation is related to the confluent hypergeometric differential equation, see 2.273(9) in reference [35]. Defining $\zeta = 2\sigma$ and $\rho_n(\zeta) = \exp(-\sigma)\psi_n(\sigma)$ equation (3.24) reduces to the confluent hypergeometric differential equation, also called Kummer's equation [35, 36]

$$\sigma\psi_n''(\sigma) + (6n-2-\sigma)\psi_n'(\sigma) - \left(3n - \frac{1}{2}\right)\psi_n\sigma) = 0. \tag{3.25}$$

The solutions of this differential equation are known. The general solution can be written in terms of Kummer's function of the first $\mathrm{M}(a,b,z)$ and second kind $\mathrm{U}(a,b,z)$ also denoted confluent hypergeometric functions of the first and second kind [36]

$$\psi_n(\sigma) = B_n \mathrm{M}\left(3n - \frac{1}{2}, 6n-2, \sigma\right) + C_n \mathrm{U}\left(3n - \frac{1}{2}, 6n-2, \sigma\right), \tag{3.26}$$

where B_n and C_n are constants to be determined by the boundary condition. Substituting all former transformations, the result for $R_n(r)$ is:

$$R_n(r) = r^{6n-3} e^{-\frac{r^2}{2}} \left[B_n \mathrm{M}\left(3n - \frac{1}{2}, 6n-2, \frac{r^2}{2}\right) + C_n \mathrm{U}\left(3n - \frac{1}{2}, 6n-2, \frac{r^2}{2}\right)\right]. \tag{3.27}$$

The particular solution we are looking for has to vanish at $r = 0$ and approach a constant value for $r \to \infty$ to obey the boundary conditions. The confluent hypergeometric function of the first kind is unity when its argument is zero, $\mathrm{M}(a, b, z = 0) = 1$, whereas the hypergeometric function of the second kind approaches infinity $\lim_{z\to 0} \mathrm{U}(a, b, z) = \lim_{z\to 0} z^{1-b} = \infty$, for b integer greater than unity [37], which is the case in our solution, $b = 6n - 2$, since n is an integer greater zero. Hence setting $C_n = 0$ the solution vanishes at $r = 0$.

Now we investigate the behaviour of our solution in the limit $r \to \infty$. The asymptote of the hypergeometric function of the first kind for large arguments $z \to \infty$ is [38]:

$$\mathrm{M}(a, b, z) \propto \frac{\Gamma(b)}{\Gamma(b-a)}(-z^{-a}) \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) + \frac{\Gamma(b)}{\Gamma(a)} e^z (z^{a-b}) \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right). \quad (3.28)$$

Hence the radial solution approaches a constant value for $r \to \infty$:

$$\lim_{r\to\infty} R_n(r) = 2^{3n-\frac{3}{2}} \frac{\Gamma(6n-2)}{\Gamma(3n-1/2)}. \quad (3.29)$$

To simplify the fitting to the boundary condition $Q(r = \infty, \varphi) = 1$ we define the constant B_n as the inverse of the limit value in equation (3.29). This way the radial part of the solution $R_n(r)$ approaches 1 in the limit of $r \to \infty$. Determining the coefficients A_n of the solution $Q(r, \varphi)$ given in equation by imposing $\Psi(\varphi) = 1$ in the limit $r \to \infty$ for $\varphi \in (\pi/6, -\pi/6)$ yields:

$$A_n = \frac{4}{\pi} \frac{1}{2n-1} \sin\left(\frac{2n-1}{2}\pi\right). \quad (3.30)$$

Finally we simplify the radial solution by use of Kummer's formula [36]:

$$e^z \mathrm{M}(a, b, -z) = \mathrm{M}(b-a, b, z) \quad (3.31)$$

Hence the solution for the infinite time survival probability of our single random walker in a wedge is in the dimensionless variables (r, φ):

$$\begin{aligned} Q(r, \varphi) &= \sum_{n=1}^{\infty} 2^{-3n+\frac{7}{2}} \frac{\Gamma(3n-1/2)}{\pi(2n-1)\Gamma(6n-2)} \sin((2n-1)\pi/2) \cos(3(2n-1)\varphi) \\ &\times r^{6n-3} \mathrm{M}\left(3n - \frac{3}{2}, 6n-2, -\frac{r^2}{2}\right). \end{aligned} \quad (3.32)$$

This sum is easily shown to converge since the summand a_n decays to zero faster than $1/n$ for $n \to \infty$. For large n the confluent hypergeometric function can be approximated by an exponential function, $\mathrm{M}\left(3n - \frac{3}{2}, 6n-2, -\frac{r^2}{2}\right) \propto \exp(-\mathrm{r}^2/4)$. The asymptotic form of the quotient of gamma functions is found to be $\Gamma(3n-1/2)/\Gamma(6n-2) \propto 2^{-3n+1}(6n-3)^{-3n+3/2}e^{3n-3/2}$ using Stirling's formula. In summary, the summand decays to zero for large n as

$$a_n \propto \frac{2^{-6n+9/2}}{(2n-1)\pi}(6n-3)^{-3n+3/2} r^{6n-3} e^{3n-3/2} e^{-r^2/4}, \tag{3.33}$$

where the oscillating sine and cosine functions have been omitted. Although the sum clearly converges, the computational equipment was not sufficient to calculate the sum in general or at specific points. Therefore, all plots of the solution to be displayed here are only approximations including the first thirty terms of the sum, which is sufficient in the chosen range, since, for instance, the error due to the absence of the next ten terms, up to term 40, is smaller than 5×10^{-37}.

To plot and analyse the infinite time survival probability we transform the solution back to the dimensionless relative coordinates $\tilde{y}_1$ and $\tilde{y}_2$ of the three vicious walkers. In those coordinates the result reads:

$$\begin{aligned} Q(\tilde{y}_1, \tilde{y}_2) &= \sum_{n=1}^{\infty} (-1)^{n+1} 2^{3n+\frac{1}{2}} \frac{\Gamma(3n-1/2)}{\pi(2n-1)\Gamma(6n-2)} \left(\frac{1}{3}(\tilde{y}_1^2 + \tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right)^{3n-3/2} \\ &\times \cos\left[3(2n-1)\arctan\left(\frac{\tilde{y}_1 - \tilde{y}_2}{\sqrt{3}(\tilde{y}_1 + \tilde{y}_2)}\right)\right] \\ &\times \mathrm{M}\left(3n - \frac{3}{2}, 6n-2, -\frac{2}{3}(\tilde{y}_1^2 + \tilde{y}_1\tilde{y}_2 + \tilde{y}_2^2)\right). \end{aligned} \tag{3.34}$$

In figure 3.4 this function is sketched in the range $\tilde{y}_1, \tilde{y}_2 \in [0, 8]$. The survival probability smoothly increases from zero at $\tilde{y}_1 = 0$ and $\tilde{y}_2 = 0$ to form a plateau of constant probability for $\tilde{y}_1 > 2$ and $\tilde{y}_2 > 2$ that increases to unity at $\tilde{y}_1 = \infty$ and $\tilde{y}_2 = \infty$, corresponding to certain survival, if all three vicious walkers start infinitely far apart. Unfortunately Mathematica could not calculate the sum for $\tilde{y}_1 \to 0$ and $\tilde{y}_2 \to 0$, but the summand of equation (3.34) clearly approaches zero for $\tilde{y}_1 \to 0$ and $\tilde{y}_2 \to 0$. This is obvious, since the hypergeometric function of the first kind is finite at zero, the cosine is always well-defined and summand is proportional to $\tilde{y}_1$ and $\tilde{y}_2$.

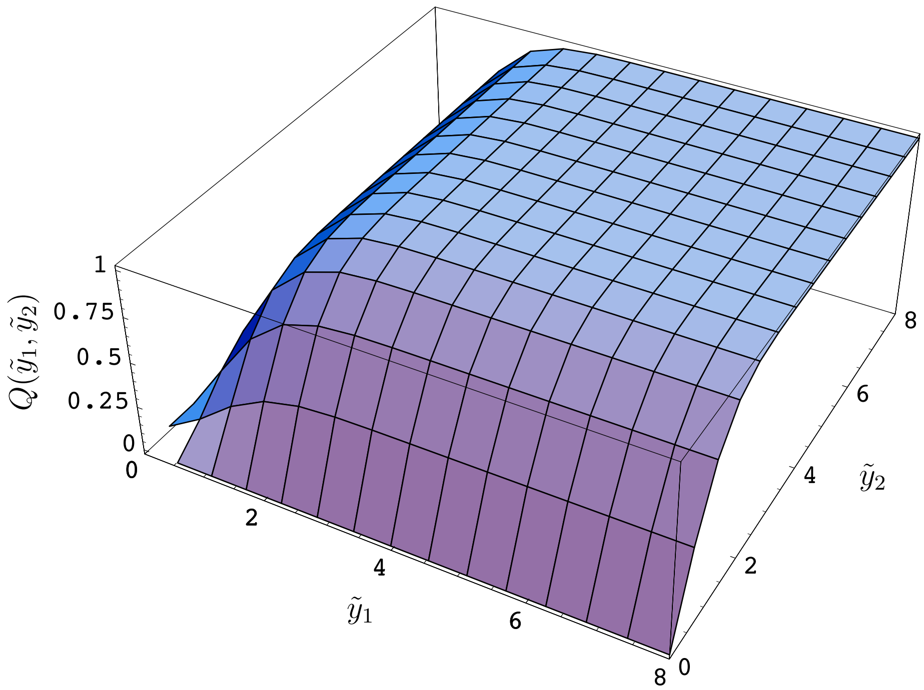

Figure 3.4: The infinite time survival probability of three vicious walkers in an inverted harmonic potential versus the dimensionless relative coordinates $\tilde{y}_1 = \sqrt{a/2D}(x_2 - x_1)$ and $\tilde{y}_2 = \sqrt{a/2D}(x_3 - x_2)$.

To study the survival probability further, it is also of interest to consider the contour lines of figure 3.4 as shown in figure 3.5.

Investigating those one easily recognises that the function is symmetric about the $\tilde{y}_1 = \tilde{y}_2$ axis, i.e. for say $\tilde{y}_1 = c$, c an arbitrary constant, $Q(\tilde{y}_1 = c, \tilde{y}_2)$ is given by the same function as $Q(\tilde{y}_1, \tilde{y}_2 = c)$.

Setting one relative coordinate to a constant value physically means keeping two of the three vicious walkers at a fixed distance to each other while the third is diffusing freely. Hence the problem is the same for fixing the distance of the first two walkers or the last two walkers, i.e. $\tilde{y}_1$ or $\tilde{y}_2$, respectively. Furthermore, in the limit of setting one relative coordinate to infinity, say $\tilde{y}_2 = \infty$, the three walkers problem simplifies to the problem of two vicious walkers in an inverted harmonic potential. The survival probability of two vicious walkers in a general harmonic potential $V(x) = \mu x^2/2$ has been calculated in section 1.4.2. Setting $\mu = -|a|$ in the solution (1.36) and taking

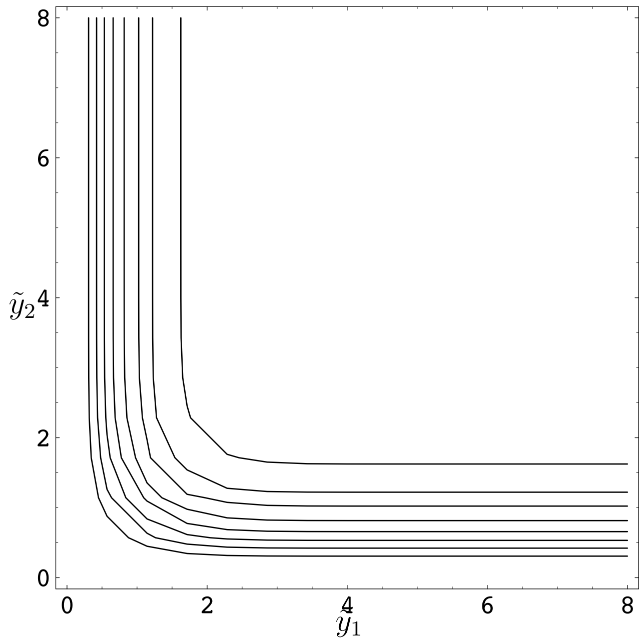

Figure 3.5: Contour lines of the infinite time survival probability of three vicious walkers in an inverted harmonic potential versus the relative coordinates $\tilde{y}_1 = \sqrt{a/2D}(x_2 - x_1)$ and $\tilde{y}_2 = \sqrt{a/2D}(x_3 - x_2)$. The different lines correspond to constant probabilities of 0.1 up to 0.8.

the limit of infinite time, the result for the survival probability of two vicious walkers is in dimensionless variables:

$$Q(\tilde{y}_1, \tilde{y}_2 = \infty) = \mathrm{Erf}(\frac{\tilde{y}_1}{\sqrt{2}}). \tag{3.35}$$

Unfortunately showing this limiting behaviour analytically has proved to be intractable. Instead we plotted $Q(\tilde{y}_1, \tilde{y}_2 = c)$ for $c = 2, 3, \ldots, 10$, see figure 3.6. The figure clearly shows how the sequence of graphs approaches the error function (red) expected for $\tilde{y}_2 = \infty$, see equation (3.35). All graphs starting from $c = 4$ lie exactly on top of the error function, proving the limiting behaviour.

3.4 Summary

In this chapter we used the interesting features of diffusion in a wedge to solve the problem of three vicious walkers in an inverted harmonic potential. In a review explaining the properties of the wedge, we derived the asymptote of the survival probability of

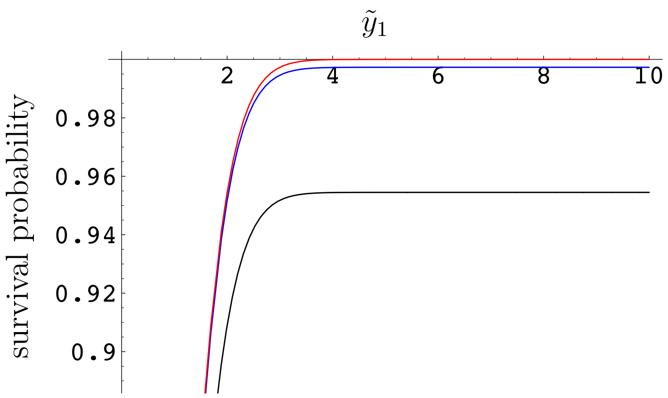

Figure 3.6: The infinite time survival probability of three vicious walkers in an inverted harmonic potential keeping the relative coordinate $\tilde{y}_2 = c$ fixed at c=2(black), c=3(blue) and c=4(red), where the red graph is already indistinguishable from the error function (3.35).

a random walker in an absorbing wedge by use of scaling properties. In addition, the problem of diffusion of three vicious walkers in one dimension has been mapped onto a diffusion process of a single random walker in a wedge. Although this mapping only enables us to solve problems with three interacting walkers, very complicated problems such as vicious walkers with different diffusion constants are solvable in a very elegant way as has been shown. Finally, the mapping has been applied to a new problem, to calculate the infinite time survival probability of three vicious walkers in an inverted harmonic potential.

Chapter 4

Conclusion

In this thesis the backward Fokker-Planck equation has been used to investigate the properties of vicious walkers especially concerning motion in one-body potentials. Exact results for the asymptotic or limiting form of the survival probability of vicious walkers have been obtained.

In chapter 2 the main interest lay in attractive one-body potentials. We introduced a method to generalise results for the survival probability of a single random walker in an arbitrary attractive potential to obtain the survival probability of N vicious walkers in such a potential. This method of antisymmetrising has been applied to obtain the asymptotic form of the survival probability of N vicious walkers in a square-well potential with absorbing or reflecting boundary conditions at the walls, and in a harmonic potential with or without an absorbing or reflecting boundary at the origin. The results for the harmonic potential have been used to calculate the problem of N vicious walkers on a line restricted by an absorbing or reflecting wall at the origin by mapping the zero potential case onto the harmonic potential.

Chapter 3 concerned three vicious walkers in an inverted harmonic potential. To solve this problem diffusion processes in a two-dimensional wedge have been investigated. By mapping the problem of three one-dimensional vicious walkers in an inverted harmonic potential onto the diffusion of a single random walker in a two-dimensional wedge, we obtained their survival probability in the limit of infinite time.

In summary, various problems concerning vicious walkers in one-body potentials have been addressed giving exact results for the time-dependence of the analysed processes and the limiting probability distribution.

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